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Symplectic q-Schur algebras

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1. Introduction

The general linear group $GL_n(K)$ operates on $V^{\otimes r}$ the r-fold tensor space of its natural module V. Its group algebra factored by the kernel of this operation is called the *Schur algebra* and denoted $S(n,r)$. By place permutation the symmetric group $\mathcal{S}_r$ operates on $V^{\otimes r}$ too. Moreover, both actions centralize each other. This fact is known as *Schur–Weyl duality*.

This situation admits a q-analogue which has been introduced by R. Dipper and G. James in [DJ]. Here, instead of the symmetric group you have to take the *Iwahori–Hecke algebra* of type A. Its centralizer is called the *q-Schur algebra*. There are various generalizations of this theory for instance by Dipper et al. [DJM] who replaced the Iwahori–Hecke algebras by *Ariki–Koike algebras* leading to so called *cyclotomic q-Schur algebras*. On the other hand the original q-Schur algebra can be obtained (up to Morita equivalence, cf. [DJ2]) using constructions from the theory of *quantum groups* [DD]. In this paper we will apply these constructions to obtain q-Schur algebras which are related to the *symplectic groups*. We will denote them by $S_q^s(n,r)$. Setting the deformation parameter $q=1$, we obtain classical symplectic Schur algebras in the sense of S. Donkin [Do1]. The main result in this paper is that the symplectic q-Schur algebras are *cellular* in the sense of J.J. Graham and G.I. Lehrer [GL] and *integrally quasi-hereditary* as algebras over the ring of integer Laurent polynomials.

In order to obtain the cellular basis we introduce a quantum symplectic version of *bideterminants*. In [O2] the author has presented a symplectic version of the famous

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straightening formula for bideterminants in the classical case. Here, we will develop the fundamental calculus for quantum symplectic bideterminants and give a quantized version of that straightening formula. This formula is powerful enough to imply almost all results of the paper.

The *standard modules* (or *cell representations*) of $S_q^s(n,r)$ are indexed by pairs (λ, l) consisting of an integer $0 \leqslant l \leqslant r/2$ and a partition $\lambda \in \Lambda^+(m, r-2l)$ of $r-2l$ into not more than m parts. Here $n = 2m$ is the dimension of the natural module of the symplectic group. The part of the basis corresponding to (λ, l) is labelled by pairs of λ-*symplectic standard tableaux* in the sense of R.C. King [Ki], or more precisely by a reversed version of them.

The material of this paper is taken from my doctoral thesis [O1] arranged in a completely reorganized form. Furthermore, it contains some improvements. Thus the restrictions in [O1, 3.12.14 and 4.1.2] have been removed in Theorems 7.1 and 7.3. The technical ingredients for this are developed in Section 14. Also, the proof of Proposition 12.1 is more direct and shortened compared to [O1, 3.10.4].

2. Quantum symplectic monoids

Let R be a noetherian integral domain and $q \in R$ an invertible element. Let V be a free R-module of rank $n = 2m$. Fix a basis $\{v_1, \ldots, v_n\}$ and let e_{ij} denote the corresponding basis of matrix units for $\mathcal{E} := \mathrm{End}_R(V)$. We will define two endomorphisms β and γ on $V \otimes V$ identifying $\mathrm{End}_R(V \otimes V)$ with $\mathcal{E} \otimes \mathcal{E}$ (we write simply $\otimes$ instead of $\otimes_R$ if no ambiguity can arise). Some additional notation is needed. We set

$$(\rho_1, \ldots, \rho_n) = \bigl(m, m-1, \ldots, 1, -1, \ldots, -(m-1), -m\bigr)$$

and $\epsilon_i := \mathrm{sign}(\rho_i)$. Further, $i' := n - i + 1$ defines an involution on $\underline{n} := \{1, \ldots, n\}$. Thus

$$(1', 2', \ldots, n') = (n, n-1, \ldots, 1).$$

The following definition is taken from [Ha2, Eqs. (4.3), (4.5)] (respectively [Ha1, Section 5]) using the transformation $\beta = q^2 \beta_{q^{-1}}(C_m)$ and $\gamma = \iota_q$:

$$\begin{aligned}\beta := \sum_{1 \leqslant i \leqslant n} \bigl(q^2 e_{ii} \otimes e_{ii} + e_{ii'} \otimes e_{i'i}\bigr) + q \sum_{1 \leqslant i \neq j,\, j' \leqslant n} e_{ij} \otimes e_{ji} \\ + \bigl(q^2 - 1\bigr) \sum_{1 \leqslant j < i \leqslant n} \bigl(e_{ii} \otimes e_{jj} - q^{\rho_i - \rho_j} \epsilon_i \epsilon_j e_{ij'} \otimes e_{i'j}\bigr),\end{aligned}$$

and

$$\gamma := \sum_{1 \leqslant i, j \leqslant n} q^{\rho_i - \rho_j} \epsilon_i \epsilon_j e_{ij'} \otimes e_{i'j}.$$

There are slightly more general versions of these endomorphisms involving additional parameters. We may omit them without loss of generality (see [O1, Satz 2.5.8]). The operators β and γ are related to each other by the equation (cf. [Ha2, Eq. (4.4)])

$$\left(q^2-1\right)(\gamma - \mathrm{id}_{V^{\otimes 2}}) = q^2\beta^{-1} - \beta. \tag{1}$$

For $r \in \mathbb{N}$ write $\underline{r} := \{1, \ldots, r\}$. A *multi-index* is a map $\mathbf{i} : \underline{r} \to \underline{n}$ frequently denoted as an r-tuple $\mathbf{i} = (i_1, \ldots, i_r)$ where $i_j \in \underline{n}$. The set of all such multi-indices will be denoted by $I(n,r)$. We define

$$v_{\mathbf{i}} := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in \underbrace{V \otimes V \otimes \cdots \otimes V}_{r \text{ times}} =: V^{\otimes r}.$$

An endomorphism μ of $V^{\otimes r}$ may be given by its coefficients $\mu_{\mathbf{ij}}$ with respect to the basis $\{v_{\mathbf{i}} \mid \mathbf{i} \in I(n,r)\}$ of $V^{\otimes r}$, that is

$$\mu(v_{\mathbf{j}}) = \sum_{\mathbf{i} \in I(n,r)} \mu_{\mathbf{ij}} v_{\mathbf{i}}.$$

Let $F_R(n) := R\langle X_{11}, X_{12}, \ldots, X_{nn}\rangle$ be the free algebra generated by the n^2 symbols X_{ij} for $i, j \in \underline{n}$. This is a graded algebra; an R-basis of the rth homogeneous part $F_R(n,r)$ is the set

$$\left\{X_{\mathbf{ij}} := X_{i_1 j_1} \cdots X_{i_2 j_2} \cdots X_{i_r j_r} \mid \mathbf{i}, \mathbf{j} \in I(n,r)\right\}.$$

To simplify notation we introduce a new convention to write down frequently used elements of $F_R(n)$ and its quotients in a convenient way. For an endomorphism μ on $V^{\otimes r}$ we write

$$\mu \wr X_{\mathbf{ij}} := \sum_{\mathbf{k} \in I(n,r)} \mu_{\mathbf{ik}} X_{\mathbf{kj}} \quad \text{and} \quad X_{\mathbf{ij}} \wr \mu := \sum_{\mathbf{k} \in I(n,r)} X_{\mathbf{ik}} \mu_{\mathbf{kj}}. \tag{2}$$

This definition can be linearly extended to all of $F_R(n,r)$. The following rules are easily checked:

$$\begin{aligned} 1 \wr X_{\mathbf{ij}} &= X_{\mathbf{ij}} = X_{\mathbf{ij}} \wr 1, \\ \mu \wr (\nu \wr X_{\mathbf{ij}}) &= (\nu\mu) \wr X_{\mathbf{ij}}, \\ (\mu \wr X_{\mathbf{ij}}) \wr \nu &= \mu \wr (X_{\mathbf{ij}} \wr \nu). \end{aligned} \tag{3}$$

We will denote the residue classes of X_{ij} in any quotient of $F_R(n)$ by x_{ij}. The residue class $x_{\mathbf{ij}}$ of $X_{\mathbf{ij}}$ then clearly has a similar expression in the x_{ij} as the $X_{\mathbf{ij}}$ do in the X_{ij}. The above introduced convention will be used for $x_{\mathbf{ij}}$ accordingly.

The object of our investigations is given by the following definition:

$$A^{\mathrm{s}}_{R,q}(n) := F_R(n) \big/ \left\langle \beta \wr X_{\mathbf{ij}} - X_{\mathbf{ij}} \wr \beta,\ \gamma \wr X_{\mathbf{ij}} - X_{\mathbf{ij}} \wr \gamma \mid \mathbf{i}, \mathbf{j} \in I(n,2)\right\rangle.$$

Here the brackets $\langle\ \rangle$ denote the ideal generated by the enclosed elements and β, γ are the endomorphisms on $V \otimes V$ defined above. Since this ideal in the definition is homogeneous, the algebra $A^{\mathrm{s}}_{R,q}(n) = \bigoplus_{r \in \mathbb{N}_0} A^{\mathrm{s}}_{R,q}(n,r)$ is again graded. Here, $A^{\mathrm{s}}_{R,q}(n,r)$ is the R-linear span of the elements $x_{\mathbf{ij}}$ for $\mathbf{i}, \mathbf{j} \in I(n,r)$. The algebra $A^{\mathrm{s}}_{R,q}(n)$ can be identified with a generalized FRT-construction with respect to the subset $N := \{\beta, \gamma\} \subseteq \mathcal{E} \otimes \mathcal{E}$ denoted $\mathcal{M}_R(N)$ in [O2, Section 5]. It has been pointed out there that it possesses the structure of a bialgebra where comultiplication and augmentation on the generators $x_{\mathbf{ij}}$ are given by

$$\Delta(x_{\mathbf{ij}}) = \sum_{\mathbf{k} \in I(n,r)} x_{\mathbf{ik}} \otimes x_{\mathbf{kj}}, \qquad \epsilon(x_{\mathbf{ij}}) = \delta_{\mathbf{ij}}. \tag{4}$$

In particular, the homogeneous summands $A^{\mathrm{s}}_{R,q}(n,r)$ are subcoalgebras. Furthermore, the tensor space $V^{\otimes r}$ is an $A^{\mathrm{s}}_{R,q}(n)$ (respectively $A^{\mathrm{s}}_{R,q}(n,r)$)-(right-)comodule. The structure map $\tau_r : V^{\otimes r} \to V^{\otimes r} \otimes A^{\mathrm{s}}_{R,q}(n,r)$ is defined by

$$\tau_r(v_{\mathbf{j}}) = \sum_{\mathbf{i} \in I(n,r)} v_{\mathbf{i}} \otimes x_{\mathbf{ij}}.$$

Now, if $q^2 - 1$ is an invertible element in R, the endomorphism γ is known to be in the algebraic span of β; explicitly one has

$$\gamma = \frac{q^2\beta^{-1} - \beta}{q^2 - 1} + \mathrm{id}_{V^{\otimes 2}}.$$

Thus, by [O2, Corollary 2.3] the relations $\gamma \wr x_{\mathbf{ij}} = x_{\mathbf{ij}} \wr \gamma$ are redundant in this case. The reader may check that under these circumstances our bialgebra $A^{\mathrm{s}}_{R,q}(n)$ is identical to the matrix bialgebra of the usual FRT-construction $F_R(n)/\langle \beta \wr X_{\mathbf{ij}} - X_{\mathbf{ij}} \wr \beta,\ \mathbf{i}, \mathbf{j} \in I(n,2)\rangle$ connected with the symplectic group for example denoted $\mathcal{F}_\beta(M_n)$ in [CP, 7.3 c].

On the other hand, if $q^2 - 1$ is not invertible we really need to add the relations $\gamma \wr x_{\mathbf{ij}} = x_{\mathbf{ij}} \wr \gamma$. For instance, it has been proved in [O2, Corollary 6.2] that, setting $q = 1$, the bialgebra $A^{\mathrm{s}}_{R,q}(n)$ is the coordinate ring of the symplectic monoid scheme $\mathrm{SpM}_n(R)$ which is defined by

$$\mathrm{SpM}_n(R) := \bigl\{A \in \mathrm{M}_n(R) \mid \exists d(A) \in R,\ A^t J A = A J A^t = d(A) J\bigr\}.$$

Here, J is the Gram matrix of the canonical skew bilinear form, that is $J = (J_{ij})_{i,j \in \underline{n}}$ where $J_{ij} := \epsilon_i \delta_{ij'}$. The regular function $d : \mathrm{SpM}_n(R) \to R$ is called the *coefficient of dilation* (cf. [Dt1]). On the other hand, in this case the bialgebra of the usual FRT-construction equals $A_R(n) = R[x_{11}, x_{12}, \ldots, x_{nn}]$, the commutative polynomial ring in the x_{ij}, which is just the coordinate ring of the monoid scheme $\mathrm{M}_n(R)$ of $(n \times n)$-matrices. Consequently the bialgebra of the usual FRT-construction contains $(q^2 - 1)$-torsion elements considered over the ground ring $R = \mathbb{Z}[q, q^{-1}]$ of integer Laurent polynomials in q.

Let us write down a couple of consequent relations holding in $A^{\mathrm{s}}_{R,q}(n)$. For this purpose the algebraic span of the $V^{\otimes r}$-endomorphisms

$$\beta_i := \mathrm{id}_{V^{\otimes i-1}} \otimes \beta \otimes \mathrm{id}_{V^{\otimes r-i-1}} \quad \text{and} \quad \gamma_i := \mathrm{id}_{V^{\otimes i-1}} \otimes \gamma \otimes \mathrm{id}_{V^{\otimes r-i-1}}, \quad i = 1, \dots, r-1,$$

in $\mathrm{End}_R(V^{\otimes r})$ will be denoted by $\mathcal{A}_r$ (for all $r > 1$). According to [O2, Sections 1, 5] in $A^{\mathrm{s}}_{R,q}(n,r)$ the following relations hold for all $r > 1$:

$$\mu \wr x_{\mathbf{ij}} = x_{\mathbf{ij}} \wr \mu \quad \text{for all } \mu \in \mathcal{A}_r,\ \mathbf{i}, \mathbf{j} \in I(n,r). \tag{5}$$

The reader should also note that by [O2, Lemma 2.2] all elements of $\mathcal{A}_r$ must be morphisms of $A^{\mathrm{s}}_{R,q}(n,r)$-comodules.

3. Quantum symplectic bideterminants

Let $p, r \in \mathbb{N}$ be positive integers and $\Lambda(p,r)$ denote the set of compositions of r into p parts. These are p-tuples $\lambda = (\lambda_1, \dots, \lambda_p)$ of non-negative integers $\lambda_i \in \mathbb{N}_0$ summing up to r. To each composition $\lambda \in \Lambda(p,r)$ there corresponds a parabolic subgroup in the symmetric group $\mathcal{S}_r$, called the *standard Young* subgroup. We will denote it by $\mathcal{S}_\lambda$. It is the subgroup fixing the sets $\{1, 2, \dots, \lambda_1\}$, $\{\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2\}, \dots$. Now, let $w \in \mathcal{S}_r$ be given by a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_t}$, where the $s_i = (i, i+1)$ are the simple transpositions. We define endomorphisms

$$\beta(w) := \beta_{i_1} \beta_{i_2} \cdots \beta_{i_t} \in \mathrm{End}_R\big(V^{\otimes r}\big) := \mathrm{End}_R\big(V^{\otimes r}\big)$$

for $r > 1$ and set $\beta(w) = \mathrm{id}_V \in \mathcal{E}$ for $r = 1$. It is easy to see that this definition is independent of the choice of the reduced expression for w since any two of them can be transformed into each other using the braid relations. But β satisfies the *quantum Yang–Baxter equation* which is just the second type braid relation

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}$$

in the case $i = 1$. The latter one obviously implies the relations for $i > 1$, whereas the first type braid relations $\beta_i \beta_j = \beta_j \beta_i$ for $|i - j| > 1$ hold trivially. Observe that

$$\beta(ww') = \beta(w)\beta(w') \quad \text{if} \quad l(ww') = l(w) + l(w'),$$

where $l(w)$ denotes the length of w, that is the number of transpositions in a reduced expression. Setting $y := q^2$ and using our notation (2) we associate a *quantum symplectic bideterminant* to each triple consisting of a composition λ of r and a pair of multi-indices $\mathbf{i}, \mathbf{j} \in I(n,r)$ by

$$t^\lambda_q(\mathbf{i} : \mathbf{j}) := \sum_{w \in \mathcal{S}_\lambda} (-y)^{-l(w)} \beta(w) \wr x_{\mathbf{ij}} = \sum_{w \in \mathcal{S}_\lambda} (-y)^{-l(w)} x_{\mathbf{ij}} \wr \beta(w). \tag{6}$$

The equality therein follows from (5) applied to $\mu = \beta(w)$. Using the abbreviation $\kappa_\lambda := \sum_{w\in\mathcal{S}_\lambda}(-y)^{-l(w)}\beta(w) \in \mathrm{End}_R(V^{\otimes r})$ we also may write $t_q^\lambda(\mathbf{i}:\mathbf{j}) = \kappa_\lambda \wr x_{\mathbf{ij}} = x_{\mathbf{ij}} \wr \kappa_\lambda$. If q is set to 1, we obtain

$$x_{\mathbf{ij}} \wr \beta(w) = x_{\mathbf{i}(\mathbf{j}w^{-1})} \quad \text{and} \quad \beta(w) \wr x_{\mathbf{ij}} = x_{(\mathbf{i}w)\mathbf{j}}$$

since then $\beta(w)_{\mathbf{kj}} = \delta_{\mathbf{kj}w^{-1}} = \delta_{\mathbf{k}w\mathbf{j}}$. Therefore, in this case our quantum symplectic bideterminants coincide with ordinary bideterminants which are defined as products of minor $(\lambda_i \times \lambda_i)$-determinants, one factor for each entry λ_i of the composition λ. According to familiar notation we write for a partition λ

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) := t_q^{\lambda'}(\mathbf{i}:\mathbf{j}).$$

By a partition we mean a composition $\lambda = (\lambda_1, \ldots, \lambda_p)$ ordered decreasingly ($\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 0$). The subset of $\Lambda(p,r)$ consisting of all partitions will be denoted by $\Lambda^+(p,r)$. By λ' we denote the dual of the partition λ, that is $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ where $s = \lambda_1$ and $\lambda'_i := |\{j \mid \lambda_j \geqslant i\}|$. Using this notation one obtains precisely the classical bideterminant $T^\lambda(\mathbf{i}:\mathbf{j})$ (as defined in [Ma, 2.4] for instance) when q is set to 1. Observe that the capital T notation is more restricted since not all compositions occur as duals of partitions. This makes it necessary to consider $t_q^\lambda(\mathbf{i}:\mathbf{j})$ as well for technical reasons.

It should be remarked that the well-known quantum determinants corresponding to the general linear groups (see, for example, [DD, 4.1.2, 4.1.7], [CP, p. 236], [Tk, p. 152], [Ha1, p. 157]) can be defined in a similar way using the quantum Yang–Baxter operator of type A instead of our β. In contrast, explicit expressions for quantum symplectic bideterminants become very complicated for $r > 2$ (apart from the case $\lambda = \alpha_r := (r) \in \Lambda^+(1,r)$ in which case the bideterminants $T_q^{\alpha_r}(\mathbf{i}:\mathbf{j})$ just are the monomials $x_{\mathbf{ij}}$). Denoting the fundamental weights by $\omega_r := (1, 1, \ldots, 1) \in \Lambda^+(r,r)$, one obtains a single $(r \times r)$-minor determinant. If $r = 2$, explicit expressions are, for example,

$$\begin{vmatrix} x_{ki} & x_{kj} \\ x_{li} & x_{lj} \end{vmatrix}_q := T_q^{\omega_2}\big((k,l):(i,j)\big) = x_{ki}x_{lj} - q^{-1}x_{kj}x_{li}$$

if $k < l$, $i < j$, $i \neq j' = n - j + 1$ and

$$\begin{vmatrix} x_{ki} & x_{ki'} \\ x_{li} & x_{li'} \end{vmatrix}_q := T_q^{\omega_2}\big((k,l):(i,i')\big) = x_{ki}x_{li'} - q^{-2}x_{ki'}x_{li} - \big(q^{-2} - 1\big)\sum_{j=1}^{i-1} q^{j-i}x_{kj'}x_{lj},$$

in the cases $k < l$, $i \leqslant m$. The calculation of $T_q^{\omega_3}((j,k,l):(i,i',i))$ for $j < k < l$, $i \leqslant m$ is really hard work. Note that such a bideterminant might be different from zero even if it contains two identical columns.

4. Quantum coefficient of dilation

In the definition of the symplectic monoid $\mathrm{SpM}_n(R)$ we have introduced a function called the *coefficient of dilation*. This is necessarily a regular function in the sense of algebraic geometry. Now we will define its quantization which will be called the *quantum coefficient of dilation*. Using notation (2) we see that

$$-q^{-\rho_k-\rho_l}\epsilon_k\epsilon_l\gamma \wr x_{(k,k')(l,l')} = q^{-\rho_l}\epsilon_l\sum_{i=1}^{n} q^{\rho_i}\epsilon_i x_{il}x_{i'l'}$$

is independent of k, whereas

$$-q^{-\rho_k-\rho_l}\epsilon_k\epsilon_l x_{(k,k')(l,l')} \wr \gamma = -q^{-\rho_k}\epsilon_k\sum_{i=1}^{n} q^{\rho_i}\epsilon_i x_{ki}x_{k'i'}$$

is independent of l. But, as $\gamma \wr x_{(k,k')(l,l')} = x_{(k,k')(l,l')} \wr \gamma$ according to (5), both expressions coincide and consequently are independent of both k and l. Thus, the element

$$d_q := -q^{-\rho_k-\rho_l}\epsilon_k\epsilon_l\gamma \wr x_{(k,k')(l,l')} = -q^{-\rho_k-\rho_l}\epsilon_k\epsilon_l x_{(k,k')(l,l')} \wr \gamma \tag{7}$$

is well defined in $A^{\mathrm{s}}_{R,q}(n)$. In fact it is a grouplike element of this bialgebra. More precisely it is the coefficient function of the one-dimensional subcomodule of $V \otimes V$ that is spanned by the tensor

$$J^* := \sum_{i=1}^{n} \epsilon_i q^{\rho_i} v_i \otimes v_{i'} \in V \otimes V.$$

To see this, note that $J^* = \gamma(-q^{-\rho_l}\epsilon_l v_l \otimes v_{l'})$ for each l and that γ is a morphism of $A^{\mathrm{s}}_{R,q}(n)$-comodules. One calculates

$$\tau_2(J^*) = \tau_2 \circ \gamma\bigl(-q^{-\rho_l}\epsilon_l v_l \otimes v_{l'}\bigr) = (\gamma \otimes \mathrm{id})\left(\sum_{i,k=1}^{n} (v_i \otimes v_k) \otimes \bigl(-q^{-\rho_l}\epsilon_l x_{(i,k)(l,l')}\bigr)\right)$$

$$= J^* \otimes q^{-\rho_l}\epsilon_l \sum_{k=1}^{n} q^{\rho_k}\epsilon_k x_{(k,k')(l,l')} = J^* \otimes d_q.$$

Remark 4.1. The element J^* coincides with ζ from [Ha1, Section 6] if q is substituted for q^{-1}. Therefore d_q is identical to the grouplike element called quad there. By [Ha1, Corollary 6.3] it is central.

Lemma 4.2. *Let $j \in \underline{m}$ and $k, l \in \underline{n}$. Then we have*

$$\sum_{i=1}^{j} q^{-i} x_{ki} x_{li'} \wr \beta = \sum_{i=1}^{j} q^{i-2j} x_{ki'} x_{li}.$$

Proof. We calculate

$$\begin{aligned}\sum_{i=1}^{j} q^{-i} x_{ki} x_{li'} \wr \beta &= \sum_{i=1}^{j} \left(q^{-i} x_{ki'} x_{li} - (y-1) \sum_{h=1}^{i-1} y^{-i} q^{h} x_{kh'} x_{lh} \right) \\ &= \sum_{i=1}^{j} q^{-i+2} x_{ki'} x_{li} - (y-1) \sum_{1 \leqslant h \leqslant i \leqslant j} y^{-i} q^{h} x_{kh'} x_{lh}.\end{aligned}$$

Since $\sum_{1 \leqslant h \leqslant i \leqslant j} y^{-i} q^{h} x_{kh'} x_{lh} = \sum_{h=1}^{j} (\sum_{i=h}^{j} y^{-i}) q^{h} x_{kh'} x_{lh}$ and $(y-1) \sum_{i=h}^{j} y^{-i} = y^{-h+1} - y^{-j}$ we obtain

$$(y-1) \sum_{1 \leqslant h \leqslant i \leqslant j} y^{-i} q^{h} x_{kh'} x_{lh} = \sum_{i=1}^{j} q^{i} \left(y^{-i+1} - y^{-j} \right) x_{ki'} x_{li},$$

where on the right-hand side the summation index h has been replaced by i again. Substituting this into the second equation of the proof leads to our claim. □

For $l = m$ we deduce the connection formula:

Proposition 4.3. *The quantum symplectic* (2×2)*-determinants are related to the coefficient of dilation by*

$$\sum_{i=1}^{m} q^{-i} T_q^{\omega_2}\big((k,l) : (i,i')\big) = \begin{cases} q^{-k} d_q, & k = l' \text{ and } k \leqslant m, \\ 0, & k \neq l'. \end{cases}$$

Proof. By definition of bideterminants and the above lemma we have

$$\begin{aligned}\sum_{i=1}^{m} q^{-i} T_q^{\omega_2}\big((k,l) : (i,i')\big) &= \sum_{i=1}^{m} q^{-i} \left(x_{ki} x_{li'} - y^{-1} x_{ki} x_{li'} \wr \beta \right) \\ &= \sum_{i=1}^{m} q^{-i} x_{ki} x_{li'} - q^{i-2(m+1)} x_{ki'} x_{li} \\ &= q^{-(m+1)} \sum_{i=1}^{n} \epsilon_i q^{\rho_i} x_{ki} x_{li'}.\end{aligned}$$

On the other hand we see

$$\begin{aligned} x_{km}x_{lm'} \wr \gamma &= -\sum_{i=1}^{n} q^{\rho_m+\rho_i}\epsilon_m\epsilon_i x_{ki}x_{li'} \\ &= -q\sum_{i=1}^{n}\epsilon_i q^{\rho_i}x_{ki}x_{li'}. \end{aligned}$$

Putting these things together we obtain

$$\sum_{i=1}^{m} q^{-i}T_q^{\omega_2}\big((k,l):(i,i')\big) = -q^{-m-2}x_{km}x_{lm'} \wr \gamma.$$

Since $x_{km}x_{lm'} \wr \gamma = \gamma \wr x_{km}x_{lm'}$ holds by (5) it follows that the expression vanishes if $l \neq k'$. In the case $l = k'$ we deduce from (7) the equation $-q^{-m-2}x_{km}x_{lm'} \wr \gamma = q^{-k}d_q$ which finishes the proof. □

5. The symplectic q-Schur algebra

Remember that $A^{\mathrm{s}}_{R,q}(n,r)$ is a coalgebra for each r. Therefore, its dual R-module inherits the structure of an R-algebra. We define

$$S^{\mathrm{s}}_{R,q}(n,r) := \mathrm{Hom}_R\big(A^{\mathrm{s}}_{R,q}(n,r), R\big)$$

and call it the *symplectic q-Schur algebra*. Two linear forms $\mu, \nu \in S^{\mathrm{s}}_{R,q}(n,r)$ are multiplied by convolution, that is

$$\mu\nu(a) := (\mu \otimes \nu) \circ \Delta(a)$$

for all $a \in A^{\mathrm{s}}_{R,q}(n,r)$. The reader may verify that one obtains the symplectic Schur algebra in the classical situation as defined in [O2]. This also is identical to the symplectic Schur algebra in the sense of S. Donkin, respectively S. Doty ([Do2], respectively [Dt1]). One aim is to show that the construction is stable under base changes and that it is a free R-module. Both facts follow when we have shown that $A^{\mathrm{s}}_{R,q}(n,r)$ is free as an R-module. Further we want to initiate the study of the representation theory of this algebra. An easy way to do this is to check that the axioms of a *cellular algebra* given by J.J. Graham and G.I. Lehrer in [GL] hold. These axioms are as follows.

Let A be an associative unital algebra over a commutative unital ring R together with a partially ordered finite set Λ and finite sets $M(\lambda)$ to each $\lambda \in \Lambda$ (the set of "λ-tableaux"). A is called a *cellular algebra* if the following properties hold:

(C1) A possesses an R-basis $\{C^{\lambda}_{S,T} \mid \lambda \in \Lambda,\ S, T \in M(\lambda)\}$.
(C2) A possesses an R-linear involution $*$ which is an algebra anti-automorphism such that ${C^{\lambda}_{S,T}}^{*} = C^{\lambda}_{T,S}$ holds for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$.

(C3) For all $a \in A$, $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ the congruence relation

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \quad \operatorname{mod} A(<\lambda)$$

holds, where the elements $r_a(S', S) \in R$ are independent of T and $A(<\lambda)$ is defined as the R-linear span of basis elements $C^{\mu}_{U,V}$ where $\mu < \lambda$ and $U, V \in M(\mu)$.

Starting with these axioms the representation theory of A is developed in [GL] along the following lines. To each $\lambda \in \Lambda$ a standard module $W(\lambda)$ is defined on a free R-basis $\{C^{\lambda}_S \mid \in M(\lambda)\}$. An element $a \in A$ acts on it via $aC^{\lambda}_S = \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S'}$. Each $W(\lambda)$ possesses a symmetric bilinear form ϕ_λ for which the formula $\phi_\lambda(a^* x, y) = \phi_\lambda(x, ay)$ is valid for all $a \in A$ and $x, y \in W(\lambda)$. In the case where R is a field and $\phi_\lambda \neq 0$, the radical of $W(\lambda)$ is the same as the radical of the bilinear form ϕ_λ. The simple head L_λ of $W(\lambda)$ then is absolutely irreducible. In this way a complete set of pairwise non-isomorphic simple A-modules $\{L_\lambda \mid \lambda \in \Lambda_0\}$ can be obtained. Here we have set $\Lambda_0 := \{\lambda \in \Lambda \mid \phi_\lambda \neq 0\}$.

Denoting the multiplicity of L_μ in $W(\lambda)$ by $d_{\lambda\mu}$ to each $\lambda \in \Lambda$ and $\mu \in \Lambda_0$ Graham and Lehrer show that $d_{\lambda\mu} = 0$ for $\lambda \leqslant \mu$ and $d_{\lambda\lambda} = 1$. To each order refining the given partial order on Λ the corresponding decomposition matrix $D = (d_{\lambda\mu})_{\lambda \in \Lambda, \mu \in \Lambda_0}$ is unitriangular. The Cartan matrix C can be calculated as $C = D^t D$. The theory also supplies a criterion to decide whether A is semisimple or quasi-hereditary. In the first case we must have $\operatorname{rad}(\phi_\lambda) = (0)$ for all $\lambda \in \Lambda$ whereas in the second case $\Lambda_0 = \Lambda$ will do.

Examples of cellular algebras are the Brauer centralizer algebras $\mathcal{B}_{R,x,r}$, Ariki–Koike–Hecke algebras, Temperley–Lieb and Jones algebras [GL]. R.M. Green [GR] constructs a q-analogue of the *codeterminant basis* (in the sense of [Gr]) for the classical Schur algebra $S_R(n, r)$ which is cellular as well. The corresponding standard modules $W(\lambda)$ are precisely the q-Weyl modules in the sense of [DJ2] (see [GR, Proposition 5.3.6]).

It should be remarked that the finiteness of Λ is not postulated in the original definition. Since this property is valid in our example we impose this restriction to avoid unnecessary trouble (cf. discussion in [KX, Section 3]).

Since we have defined the symplectic q-Schur algebra as the dual module of a coalgebra we now translate the concept of cellular algebras to coalgebras.

Let K be a coalgebra over a commutative unital ring R, together with a partially ordered finite set Λ and finite sets $M(\lambda)$ for each $\lambda \in \Lambda$. We call K a *cellular coalgebra* if the following properties hold:

(C1*) K possesses an R-basis $\{D^{\lambda}_{S,T} \mid \lambda \in \Lambda,\ S, T \in M(\lambda)\}$.
(C2*) K possesses an R-linear involution * which is a coalgebra anti-automorphism, such that ${D^{\lambda}_{S,T}}^* = D^{\lambda}_{T,S}$ holds for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$.
(C3*) For all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ the congruence relation

$$\Delta\big(D^{\lambda}_{S,T}\big) \equiv \sum_{S' \in M(\lambda)} h(S', S) \otimes D^{\lambda}_{S',T} \quad \operatorname{mod} K \otimes K(>\lambda)$$

holds, where the coalgebra elements $h(S', S) \in K$ are independent of T and $K(>\lambda)$ is defined as the R-linear span of basis elements $D^{\mu}_{U,V}$ where $\mu > \lambda$ and $U, V \in M(\mu)$.

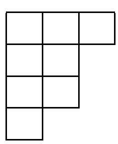
To an arbitrary R-coalgebra the dual algebra is well defined. The dual coalgebra of an algebra A is well defined if the algebra is known to be projective as an R-module, since then $(A \otimes A)^* \simeq A^* \otimes A^*$. In the case of a cellular algebra this is obviously valid. The connection between the above two concepts is given by the following proposition which can be proved straightforwardly using structure constants with respect to the bases (cf. [O1, 4.2.3]).

Proposition 5.1. *The dual algebra of a cellular coalgebra is a cellular algebra. The dual coalgebra of a cellular algebra is a cellular coalgebra. In both cases the corresponding bases and involution maps can be constructed dual to each other, i.e. in the former case $C^{\lambda}_{S,T}(D^{\mu}_{U,V})$ is 1 if $\lambda = \mu$, $S = U$ and $T = V$ but* 0 *otherwise and $C^{\lambda}_{S,T}({D^{\mu}_{U,V}}^*) = {C^{\lambda}_{S,T}}^*(D^{\mu}_{U,V})$.*

According to the proposition our next task is to find a cellular basis for the coalgebra $A^{\mathrm{s}}_{R,q}(n,r)$ together with an appropriate involution map such that the axioms of the cellular coalgebra hold. As soon as this is done the representation theory of $S^{\mathrm{s}}_{R,q}(n,r)$ is developed to the extent indicated above.

6. Tableaux

We will define a basis for $A^{\mathrm{s}}_{R,q}(n,r)$ consisting of quantum symplectic bideterminants and powers of the quantum symplectic coefficient of dilation. Since they are too large in number we have to single out an appropriate subset. This can be done using so called λ-tableaux which will be defined now. To each partition one associates a *Young*-diagram reading row lengths out of the components λ_i. For example,



is associated to $\lambda = (3,2,2,1) \in \Lambda^+(4,8)$. A λ-tableau $T^{\lambda}_{\mathbf{i}}$ is constructed from the diagram of λ by inserting the components of a multi-index $\mathbf{i} \in I(n,r)$ column by column into the boxes. In the above example:

$$T^{\lambda}_{\mathbf{i}} := \begin{array}{|c|c|c|} \hline i_1 & i_5 & i_8 \\ \hline i_2 & i_6 & \multicolumn{1}{c}{} \\ \cline{1-2} i_3 & i_7 & \multicolumn{1}{c}{} \\ \cline{1-2} i_4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

If λ is fixed we will sometimes identify multi-indices with their tableaux. We put a new order $\prec$ on the set $\underline{n}$, namely

$$m \prec m' \prec (m-1) \prec (m-1)' \prec \cdots \prec 1 \prec 1'.$$

The reason, why we prefer $\prec$ instead of the order $\ll$ considered in [O2] will become clear later on. Now, a multi-index $\mathbf{i}$ is called λ-*column standard* if the entries in $T_{\mathbf{i}}^{\lambda}$ are strictly increasing down columns according to this order. It is called λ-*row standard* if the entries are weakly increasing along rows and λ-*standard* if it is both at the same time. We write I_λ to denote the subset of $I(n,r)$ consisting of all λ-standard multi-indices. Such a multi-index $\mathbf{i} \in I_\lambda$ is called λ-*reverse symplectic standard* if for each index $i \in \underline{m}$ the occurrences of i as well as i' in $T_{\mathbf{i}}^{\lambda}$ are limited to the first $m-i+1$ rows. The corresponding subset of I_λ will be denoted by I_λ^{mys}. It can be shown that even though this set is different from the one of λ-symplectic standard tableaux (as defined in [Ki] and denoted I_λ^{sym} in [O2]), it has the same number of elements. For let $\sigma \in \mathcal{S}_n$ be the permutation transforming the order $\ll$ into $\prec$, that is $\sigma(i) := (m-i+1)'$ for $i \leqslant m$ and $\sigma(i) := m-i'+1$ for $i > m$. Then there is an induced bijection on $I(n,r)$ sending $(i_1,\dots,i_r)$ to $(\sigma(i_1),\dots,\sigma(i_r))$ which carries the set of λ-symplectic standard tableaux precisely to the set of λ-reverse symplectic standard tableaux.

Here are some examples in the case $m=3$ ($1'=6$, $2'=5$, $3'=4$):

$$\begin{array}{lll} 3 & 2 & 2 \\ 2' & 2' & \\ 2 & 1' & \\ 1 & & \end{array} \qquad \begin{array}{lll} 1' & 2' & 2' \\ 2 & 2 & \\ 2' & 3 & \\ 3' & & \end{array} \qquad \begin{array}{lll} 3 & 2 & 2 \\ 3' & 2' & \\ 1 & 1' & \\ 2' & & \end{array}$$

The first tableau is an element of I_λ^{mys} whereas the third is not. The second tableau is an element of I_λ^{sym}. It is obtained from the first one via the bijection induced from the permutation σ described above.

7. Results

Let us first describe what we will take for the set Λ occurring in the definition of the cellular coalgebra:

$$\Lambda := \left\{ \underline{\lambda} := (\lambda, l) \;\middle|\; 0 \leqslant l \leqslant \frac{r}{2},\ \lambda \in \Lambda^+(m, r-2l) \right\}.$$

According to the definition of a cellular coalgebra to each $\underline{\lambda} = (\lambda, l) \in \Lambda$ a set $M(\underline{\lambda})$ must be assigned. We take:

$$M(\underline{\lambda}) := I_\lambda^{\mathrm{mys}}.$$

Finally the basis elements themselves are defined by

$$D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}} := d_q^l T_q^\lambda(\mathbf{i}:\mathbf{j}).$$

Now, our principal aim is to prove the following

Theorem 7.1. *The R-module $A^{\mathrm{s}}_{R,q}(n,r)$ has a basis given by*

$$\mathbf{B}_r := \left\{ D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}} \mid \underline{\lambda} \in \Lambda,\ \mathbf{i},\mathbf{j} \in M(\underline{\lambda}) \right\}.$$

Furthermore, the unique R-linear map * with $D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}}{}^* = D^{\underline{\lambda}}_{\mathbf{j},\mathbf{i}}$ is an involutory coalgebra anti-automorphism and the axioms of a cellular coalgebra are satisfied.

By Proposition 5.1 we may conclude immediately:

Theorem 7.2. *The symplectic q-Schur algebra $S^{\mathrm{s}}_{R,q}(n,r)$ is a cellular algebra with the basis dual to $\mathbf{B}_r$ as a cellular basis.*

Theorem 7.3. *The symplectic q-Schur algebra is stable under base change and it is identical with the centralizer of the algebraic span $\mathcal{A}_r$ of the endomorphisms β_i and γ_i.*

Proof. This is a consequence of Theorem 7.1 by [O2, Theorems 3.3 and 4.3] (cf. [O2, Corollary 6.3]). □

Remark 7.4. The theorem sets $S^{\mathrm{s}}_{R,q}(n,r)$ into relation with the Birman–Murakami–Wenzl algebra since β_i and γ_i define a representation of it on $V^{\otimes r}$ (cf. [O1, Satz 2.2.3]).

At the end of this paper we will improve Theorem 7.2 by showing that the bilinear form ϕ_λ on the standard modules W_λ is non-zero for each λ. By [GL, 3.10] this implies

Theorem 7.5. *The symplectic q-Schur algebra $S^{\mathrm{s}}_{R,q}(n,r)$ is integrally quasi-hereditary.*

Let us first see how the involution of Theorem 7.1 arises. It realizes matrix transposition for our quantum monoid. On the generators $x_{\mathbf{ij}}$ this transposition map is defined as in the classical case by $x_{\mathbf{ij}}{}^* := x_{\mathbf{ji}}$. Indeed, this gives a well-defined algebra map on $A^{\mathrm{s}}_{R,q}(n)$, since the coefficient matrices of β and γ are symmetric (i.e. $\beta_{\mathbf{ij}} = \beta_{\mathbf{ji}}$ and $\gamma_{\mathbf{ij}} = \gamma_{\mathbf{ji}}$) implying $(\beta \wr x_{\mathbf{ij}})^* = x_{\mathbf{ji}} \wr \beta$ and $(\gamma \wr x_{\mathbf{ij}})^* = x_{\mathbf{ji}} \wr \gamma$ and thus keeping the relations of that algebra fixed. Furthermore, the endomorphisms $\kappa_\lambda \in \mathrm{End}_R(V^{\otimes r})$ must have symmetric coefficient matrices as well. We calculate

$$t_q^\lambda(\mathbf{i}:\mathbf{j})^* = (\kappa_\lambda \wr x_{\mathbf{ij}})^* = x_{\mathbf{ji}} \wr \kappa_\lambda = t_q^\lambda(\mathbf{j}:\mathbf{i}) \tag{8}$$

and in a similar way $d_q{}^* = d_q$ holds by definition (7). This shows that * factors to an algebra map of $A^{\mathrm{s}}_{R,q}(n)$. From the comultiplication rule (4) it directly follows that * is an anti-coalgebra map. This implies axiom (C2*) of a cellular coalgebra.

The verification of axiom (C3*) is the second easiest step in the proof of Theorem 7.1, but we will give it at the end of the paper since some additional ingredients are needed. The first statement of this theorem, which is axiom (C1*), is the really hard one. It is the q-analogue of [O2, Theorem 6.1]. To prove it we will proceed in a similar way as there. The difficulty is to show that $\mathbf{B}_r$ is a set of generators. For that purpose the most important step is a quantum symplectic version of the famous straightening formula.

8. The quantum symplectic straightening formula

In the classical case symplectic versions of the straightening formula have already been given in [Co, 2.4] and [O2, Section 7]. In principle, we will follow the lines of the latter paper. But there are a lot of additional difficulties, one of which forces us to work with a reversed version of λ-symplectic standard tableaux. To prepare for the statement, we define the algebra

$$A^{\mathrm{sh}}_{R,q}(n) := A^{\mathrm{s}}_{R,q}(n)/\langle d_q\rangle$$

by factoring out the ideal generated by the quantum coefficient of dilation. Since d_q is homogeneous this algebra is again graded. Let us abbreviate its rth homogeneous summand by

$$\mathcal{K} := A^{\mathrm{sh}}_{R,q}(n, r).$$

Since d_q is grouplike the comultiplication Δ obviously factors to $A^{\mathrm{sh}}_{R,q}(n)$ and $A^{\mathrm{sh}}_{R,q}(n, r)$. But $A^{\mathrm{sh}}_{R,q}(n)$ is not a bialgebra and $A^{\mathrm{sh}}_{R,q}(n, r)$ are not coalgebras, because the augmentation map ϵ does not factor. In the classical case if $R = K$ is a field $A^{\mathrm{sh}}_{K,q}(n)$ equals the coordinate ring of the symplectic semigroup $\mathrm{SpH}_n(K) := \mathrm{SpM}_n(K) \setminus \mathrm{GSp}_n(K)$ by [O2, Remark 7.5]. The missing augmentation map corresponds to the missing unit element in the semigroup.

Definition 8.1. Let A be an unital algebra and $\Delta : A \to A \otimes A$ a morphism of algebras. If Δ possesses the properties of a comultiplication we call A a semibialgebra.

By the above explanations $A^{\mathrm{sh}}_{R,q}(n)$ is a semibialgebra.

We put an order on the set $\Lambda^+(r)$ of all partitions of r, writing $\lambda < \mu$ if and only if λ' occurs before μ' in the lexicographic order. In this order the fundamental weight $\omega_r := (1, 1, \dots, 1) \in \Lambda^+(r, r)$ is the largest element, whereas $\alpha_r := (r) \in \Lambda^+(1, r)$ is the smallest one. We define $\mathcal{K}(> \lambda)$ (respectively $\mathcal{K}(\geqslant \lambda)$) to be the R-linear span in $\mathcal{K}$ of all bideterminants $T^{\mu}_q(\mathbf{i} : \mathbf{j})$ such that $\mu > \lambda$ (respectively $\mu \geqslant \lambda$) (cf. axiom (C3*) of a cellular coalgebra). Clearly $\mathcal{K} = \mathcal{K}(\geqslant \alpha_r)$.

Proposition 8.2 *(Quantum symplectic straightening formula). Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n, r)$. Then, to each $\mathbf{k} \in I^{\mathrm{mys}}_{\lambda}$ there is an element $a_{\mathbf{jk}} \in R$, such that in $\mathcal{K}$ we* *have for all* $\mathbf{i} \in I(n,r)$:

$$T_q^{\lambda}(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k}\in I_{\lambda}^{\mathrm{mys}}} a_{\mathbf{jk}} T_q^{\lambda}(\mathbf{i}:\mathbf{k}) \quad \bmod \mathcal{K}(>\lambda).$$

Before starting to prove this, we deduce its most important consequence:

Corollary 8.3. *The set* $\mathbf{B}_r$ *generates* $A^{\mathrm{s}}_{R,q}(n,r)$.

Proof. From the fact that d_q is central in $A^{\mathrm{s}}_{R,q}(n)$ by Remark 4.1 we see that multiplication by d_q from the right (written as $\cdot d_q$ below) leads to an exact sequence

$$A^{\mathrm{s}}_{R,q}(n,r-2) \xrightarrow{\cdot d_q} A^{\mathrm{s}}_{R,q}(n,r) \to A^{\mathrm{sh}}_{R,q}(n,r) \to 0 \tag{9}$$

for $r>1$. Therefore, using induction on r we can reduce to showing that

$$\left\{T_q^{\lambda}(\mathbf{i}:\mathbf{j}) \mid \lambda \in \Lambda^{+}(m,r),\ \mathbf{i},\mathbf{j} \in I_{\lambda}^{\mathrm{mys}}\right\}$$

is a set of generators for $\mathcal{K} = A^{\mathrm{sh}}_{R,q}(n,r)$. For this claim it is enough to show that

$$\mathbf{B}_{\lambda} := \left\{T_q^{\lambda}(\mathbf{i}:\mathbf{j}) \mid \mathbf{i},\mathbf{j} \in I_{\lambda}^{\mathrm{mys}}\right\}$$

is a set of generators of $\mathcal{K}(\geqslant\lambda)/\mathcal{K}(>\lambda)$ for each partition λ. To get the last claim from the straightening formula (8.2), observe that the involution * is well defined on $A^{\mathrm{sh}}_{R,q}(n)$ since $d_q{}^* = d_q$ (see Section 7). Applying * to the congruence relation of Proposition 8.2, one obtains another such formula in which the roles of $\mathbf{i}$ and $\mathbf{j}$ are exchanged. This shows that $\mathbf{B}_{\lambda}$ is indeed a set of generators for $\mathcal{K}(>\lambda)/\mathcal{K}(\geqslant\lambda)$. $\square$

In order to prove the quantum symplectic straightening formula we need a corresponding algorithm. Its classical counterpart is [O2, Proposition 7.3]. We define a map $f: I(n,r) \to \mathbb{N}_0^m$ by $f(\mathbf{i}) = (a_1,\dots,a_m)$, where

$$a_l := \left|\left\{j \in \underline{r} \mid i_j = l \text{ or } i_j = l'\right\}\right|,$$

and order $\mathbb{N}_0^m$ writing $(a_1,\dots,a_m) < (b_1,\dots,b_m)$ if and only if $(b_1,b_2,\dots,b_m)$ appears before $(a_1,a_2,\dots,a_m)$ in the lexicographic order (induced by the ordinary order on $\mathbb{N}$). Next, we obtain an order $\lhd$ on $\mathbb{N}_0^m \times I(n,r)$ defined by:

$$(a,\mathbf{i}) \lhd (b,\mathbf{j}) \quad :\Leftrightarrow \quad a<b \quad \text{or} \quad (a=b \text{ and } \mathbf{i} \prec \mathbf{j}).$$

Here, we have denoted by $\prec$ the lexicographic order on $I(n,r)$ induced by our special order $\prec$ on $\underline{n}$. Finally, we obtain a second order $\lhd$ on $I(n,r)$ via the embedding $I(n,r) \hookrightarrow \mathbb{N}_0^m \times I(n,r)$ given by $\mathbf{i} \mapsto (f(\mathbf{i}),\mathbf{i})$. Now we are able to state the symplectic straightening algorithm.

Proposition 8.4 *(Strong quantum symplectic straightening algorithm). Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n,r) \setminus I_\lambda^{\mathrm{mys}}$. Then to each $\mathbf{k} \in I(n,r)$ satisfying $\mathbf{k} \lhd \mathbf{j}$ there is an element $a_{\mathbf{jk}} \in R$ such that in $\mathcal{K}$ the following congruence relation holds for all $\mathbf{i} \in I(n,r)$:*

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k} \lhd \mathbf{j}} a_{\mathbf{jk}} T_q^\lambda(\mathbf{i}:\mathbf{k}) \quad \bmod \mathcal{K}(> \lambda).$$

Clearly, the straightening formula (8.2) is an easy consequence of the above proposition since the set $I(n,r)$ is finite and therefore the elimination of multi-indices $\mathbf{j}$ that are not λ-reverse symplectic standard in an expression $T_q^\lambda(\mathbf{i}:\mathbf{j})$ must terminate.

The proof of the straightening algorithm will take several sections. In principle we will proceed in a similar way as in [O2] to prove this algorithm, but complications arise because the embedding of the symplectic group into the general linear group does not extend to quantum groups. Instead of [O2, Proposition 7.2] we have to establish a weak form of the quantum symplectic straightening algorithm in a first step. More precisely, we will first prove Proposition 8.4 where I_λ^{mys} is substituted by I_λ. We start with some technical tools.

9. Arithmetic of bideterminants

The calculus of bideterminants is needed inside $\mathcal{K} = A_{R,q}^{\mathrm{sh}}(n,r)$. Unless otherwise stated the rules hold in $A_{R,q}^{\mathrm{s}}(n,r)$ too. Recall the definition of κ_λ from Section 3.

Lemma 9.1. *Let $\lambda \in \Lambda(p,r)$ be a composition and $y = q^2$. Then to each $i < r$ such that the simple transposition $s_i = (i, i+1)$ is contained in the standard Young-subgroup $\mathcal{S}_\lambda$, there are endomorphisms $\mu_{\lambda,i}, \mu'_{\lambda,i} \in \mathrm{End}_R(V^{\otimes r})$ satisfying*

$$\kappa_\lambda = \left(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_i\right)\mu_{\lambda,i} = \mu'_{\lambda,i}\left(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_i\right).$$

Proof. Let us first reduce to the case $\lambda = \alpha_r = (r)$. Setting $\kappa_r := \kappa_{\alpha_r}$, $k_s := \lambda_1 + \cdots + \lambda_{s-1}$, $\mu_{r,i} := \mu_{\alpha_r,i}$, $\mu'_{r,i} := \mu'_{\alpha_r,i}$ and

$$\kappa_\lambda^s := \mathrm{id}_{V^{\otimes k_s}} \otimes \kappa_{\lambda_s} \otimes \mathrm{id}_{V^{\otimes r-\lambda_s-k_s}},$$

we can extend the definition to arbitrary λ by using the formula $\kappa_\lambda = \kappa_\lambda^1 \kappa_\lambda^2 \dots \kappa_\lambda^p$ in which the factors commute. Now, using standard reduced expressions for permutations $w \in \mathcal{S}_r$ one easily verifies the following recursion rules for $r > 1$:

$$\begin{aligned}\kappa_r &= \kappa_{r-1}\left(\mathrm{id}_{V^{\otimes r}} + \sum_{l=1}^{r-1}(-y)^{l-r}\beta_{r-1}\beta_{r-2}\dots\beta_l\right) \\ &= \left(\mathrm{id}_{V^{\otimes r}} + \sum_{l=1}^{r-1}(-y)^{l-r}\beta_l\beta_{l+1}\dots\beta_{r-1}\right)\kappa_{r-1}.\end{aligned}$$

We proceed by induction on r, the case $r = 2$ being clear. The case $i < r - 1$ can be handled immediately with the help of the above recursion formula. If $i = r - 1$ we calculate

$$\kappa_r = \kappa_{r-1}\bigl(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_{r-1}\bigr) + \mu'_{r-1,r-2}\bigl(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_{r-2}\bigr)\sum_{l=1}^{r-2}(-y)^{l-r}\beta_{r-1}\beta_{r-2}\dots\beta_l.$$

But by the braid relations we get

$$\bigl(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_{r-2}\bigr)(-y)^{l-r}\beta_{r-1}\beta_{r-2}\dots\beta_l = (-y)^{l-r}\beta_{r-1}\beta_{r-2}\dots\beta_l\bigl(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_{r-1}\bigr),$$

yielding the right-hand side factorization of κ_r. The other formula is obtained similarly. □

Corollary 9.2. *Let $\mathbf{j} \in I(n,r)$ be a multi-index possessing two identical neighbouring indices $j_l = j_{l+1}$ and $\lambda \in \Lambda(p,r)$ such that the transposition s_l is contained in $\mathcal{S}_\lambda$. Then $t_q^\lambda(\mathbf{i}:\mathbf{j}) = t_q^\lambda(\mathbf{j}:\mathbf{i}) = 0$ holds for all $\mathbf{i} \in I(n,r)$.*

Proof. By assumption, $v_{\mathbf{j}}$ lies in the kernel of $(\mathrm{id}_{V^{\otimes r}} - y^{-1}\beta_l)$. Consequently the assertion concerning $t_q^\lambda(\mathbf{i}:\mathbf{j})$ follows immediately from Lemma 9.1 since $t_q^\lambda(\mathbf{i}:\mathbf{j}) = x_{\mathbf{ij}} \wr \kappa_\lambda$. Using the matrix transposition map * introduced in Section 8, the formula for exchanged multi-indices follows as well. □

Next, we investigate the transition from $A^{\mathrm{s}}_{R,q}(n)$ to its epimorphic image $A^{\mathrm{sh}}_{R,q}(n)$. For this purpose denote by $\mathcal{G}_r$ the ideal generated by $G := \gamma_1 = \gamma \otimes \mathrm{id}_{V^{\otimes r-2}}$ in the algebraic span $\mathcal{A}_r$ of the endomorphisms β_i and γ_i. By Eq. (1) the relation $\beta^2 = (q^2 - 1)\beta + q^2\,\mathrm{id}_{V^{\otimes r}2}$ holds in $\mathcal{A}_r/\mathcal{G}_r$. By the braid relations $\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$ the relations $\beta_i^2 = (q^2 - 1)\beta_i + q^2\,\mathrm{id}_{V^{\otimes r}}$ and $\gamma_i = 0$ must hold in $\mathcal{A}_r/\mathcal{G}_r$ for all i as well. The Iwahori–Hecke algebra $\mathcal{H}_q(r)$ of type A is defined on generators T_{s_i} for $i \in \{1,\dots,r-1\}$ by relations

$$T_{s_i}T_{s_j} = T_{s_j}T_{s_i} \quad \text{where } |i-j| > 1,$$
$$T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \quad \text{where } i < r-1,$$
$$T_{s_i}^2 = \bigl(q^2 - 1\bigr)T_{s_1} + q^2.$$

Therefore, there is an epimorphism from $\mathcal{H}_q(r)$ to the quotient $\mathcal{A}_r/\mathcal{G}_r$ sending the generator T_{s_l} to $\beta_l + \mathcal{G}_r$ (notation as in [DD]).

Lemma 9.3. *Let $A, B \in \mathcal{A}_r$ be endomorphisms of $V^{\otimes r}$ such that $A \equiv B$ modulo $\mathcal{G}_r$. Then, the equation $x_{\mathbf{ij}} \wr A = x_{\mathbf{ij}} \wr B$ holds in $\mathcal{K}$.*

Proof. We have to show that $x_{\mathbf{ij}} \wr A = 0$ for all $A \in \mathcal{G}_r$. Let $F, H \in \mathcal{A}_r$ be such that $A = FGH$. From the defining equation of the quantum coefficient of dilation d_q from Section 4 we have

$$x_{\mathbf{ij}} \wr G = \begin{cases} 0, & j'_1 \neq j_2 \text{ or } i'_1 \neq i_2, \\ -q^{\rho_{i_1}+\rho_{j_1}}\epsilon_{i_1}\epsilon_{j_1}d_q x_{i_3 j_3}\dots x_{i_r j_r}, & j'_1 = j_2 \text{ and } i'_1 = i_2. \end{cases}$$

This means $x_{\mathbf{ij}} \wr G = 0$ in $\mathcal{K}$ for all $\mathbf{i}, \mathbf{j} \in I(n,r)$. By (5) we have $x_{\mathbf{ij}} \wr F = F \wr x_{\mathbf{ij}}$ and therefore,

$$\begin{aligned} x_{\mathbf{ij}} \wr FGH &= \sum_{\mathbf{k},\mathbf{l},\mathbf{s}\in I(n,r)} x_{\mathbf{ik}} f_{\mathbf{kl}} g_{\mathbf{ls}} h_{\mathbf{sj}} = \sum_{\mathbf{k},\mathbf{l},\mathbf{s}\in I(n,r)} f_{\mathbf{ik}} x_{\mathbf{kl}} g_{\mathbf{ls}} h_{\mathbf{sj}} \\ &= \sum_{\mathbf{k},\mathbf{s}\in I(n,r)} f_{\mathbf{ik}} (x_{\mathbf{ks}} \wr G) h_{\mathbf{sj}} = 0, \end{aligned}$$

where $(f_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I(n,r)}$, $(g_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I(n,r)}$ and $(h_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I(n,r)}$ are the coefficient matrices of F, G, and H. $\square$

We extend the notation introduced in (2). Let $\mu \in \mathrm{End}_R(V^{\otimes r})$ be an endomorphism of $V^{\otimes r}$. Set

$$\begin{aligned} t_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \mu &:= \sum_{\mathbf{k}\in I(n,r)} t_q^\lambda(\mathbf{i}:\mathbf{k}) \mu_{\mathbf{kj}} = x_{\mathbf{ij}} \wr (\kappa_\lambda \mu), \\ \mu \wr t_q^\lambda(\mathbf{i}:\mathbf{j}) &:= \sum_{\mathbf{k}\in I(n,r)} \mu_{\mathbf{ik}} t_q^\lambda(\mathbf{k}:\mathbf{j}) = (\mu\kappa_\lambda) \wr x_{\mathbf{ij}}. \end{aligned} \tag{10}$$

Similar expressions are used with respect to the capital T notation for bideterminants.

Lemma 9.4. *For all $\mathbf{i}, \mathbf{j} \in I(n,r)$ and $w \in \mathcal{S}_\lambda$ the following equations hold in $\mathcal{K}$:*

$$\beta(w) \wr t_q^\lambda(\mathbf{i}:\mathbf{j}) = \beta(w)^{-1} \wr t_q^\lambda(\mathbf{i}:\mathbf{j}) = (-1)^{l(w)} t_q^\lambda(\mathbf{i}:\mathbf{j}) = t_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \beta(w)^{-1} = t_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \beta(w).$$

Proof. Modulo $\mathcal{G}_r$ we have

$$\beta(w)\kappa_\lambda = \beta(w)^{-1}\kappa_\lambda = (-1)^{l(w)}\kappa_\lambda = \kappa_\lambda\beta(w)^{-1} = \kappa_\lambda\beta(w)$$

since the corresponding equations (where $\beta(w)$ is replaced by T_w) hold in the Iwahori–Hecke algebra $\mathcal{H}_q(r)$. Thus the assertion follows from Lemma 9.3. $\square$

Let $\mathcal{J}$ denote the ideal in the tensor algebra $\mathcal{T}(V) = \bigoplus_{r\in\mathbb{N}_0} V^{\otimes r}$ generated by the twofold invariant tensor $J^* = \sum_{i=1}^n \epsilon_i q^{\rho_i} v_i \otimes v_{i'} \in V \otimes V$ and let $\mathcal{J}_r := \mathcal{J} \cap V^{\otimes r}$ be its rth homogeneous summand.

Lemma 9.5. *Let U be the R-linear span of all elements $\gamma_l(v_\mathbf{i})$ where $1 \leqslant l < r$ and $\mathbf{i} \in I(n,r)$. Then $U = \mathcal{J}_r$.*

Proof. Since $\gamma(v_k v_{k'}) = -\epsilon_k q^{\rho_k} J^*$ and $\gamma(v_k v_l) = 0$ for all $k, l \in \underline{r}$ with $k \neq l'$ by Section 4 it follows that U is contained in $\mathcal{J}_r$. The verification of the opposite inclusion can be reduced to consider elements of the form $v_\mathbf{i} J^* v_\mathbf{j}$ with $\mathbf{i} \in I(n, l-1)$, $\mathbf{j} \in I(n, r-l-1)$ for some $1 \leqslant l < r$. But such an element can be written as $-\epsilon_k q^{-\rho_k} \gamma_l(v_\mathbf{i} v_k v_{k'} v_\mathbf{j})$ for some $k \in \underline{r}$. Thus the assertion follows. $\square$

Lemma 9.6. *Let $a_{\mathbf{j}} \in R$ be such that $\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} v_{\mathbf{j}} \in \mathcal{J}_r$. Then, for all $\mathbf{i} \in I(n,r)$ and all compositions λ of r we have in $\mathcal{K}$*

$$\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} x_{\mathbf{ij}} = 0 \quad \textit{and} \quad \sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} t_q^{\lambda}(\mathbf{i}:\mathbf{j}) = 0.$$

Proof. First, note that the second equation follows from the first one by definition of bideterminants. By the above lemma we can reduce to the case $\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} v_{\mathbf{j}} = \gamma_l(v_{\mathbf{k}})$ where $\mathbf{k} \in I(n,r)$, $1 \leqslant l < r$ and $a_{\mathbf{j}} = (\gamma_l)_{\mathbf{jk}}$. Thus we get $\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} x_{\mathbf{ij}} = x_{\mathbf{ik}} \wr \gamma_l = 0$. □

Next, we give a quantum symplectic version of *Laplace duality*. The corresponding classical result can be found in [Ma, 2.5.1], for instance.

Proposition 9.7 *(Laplace duality). Let $\lambda, \mu \in \Lambda(p,r)$ be compositions, Y a set of left coset representatives of $\mathcal{S}_\lambda \cap \mathcal{S}_\mu$ in $\mathcal{S}_\lambda$ and X a set of right coset representatives of $\mathcal{S}_\lambda \cap \mathcal{S}_\mu$ in $\mathcal{S}_\mu$, such that $l(vw) = l(v) + l(w)$ and $l(wu) = l(u) + l(w)$ holds for all $v \in Y$, $u \in X$ and $w \in \mathcal{S}_\lambda \cap \mathcal{S}_\mu$. Then for all $\mathbf{i}, \mathbf{j} \in I(n,r)$ the following equation holds*:

$$\sum_{u\in X} (-y)^{-l(u)} t_q^{\lambda}(\mathbf{i}:\mathbf{j}) \wr \beta(u) = \sum_{v\in Y} (-y)^{-l(v)} \beta(v) \wr t_q^{\mu}(\mathbf{i}:\mathbf{j}).$$

Proof. Using the fact $\beta(w)\beta(u) = \beta(wu)$, $\beta(v)\beta(w) = \beta(vw)$ which holds by length additivity we calculate

$$\begin{aligned}
\sum_{u\in X} (-y)^{-l(u)} t_q^{\lambda}(\mathbf{i}:\mathbf{j}) \wr \beta(u) &= \sum_{w'\in\mathcal{S}_\lambda} \sum_{u\in X} (-y)^{-l(w')-l(u)} x_{\mathbf{ij}} \wr \beta(w')\beta(u) \\
&= \sum_{v\in Y} \sum_{w\in\mathcal{S}_\lambda\cap\mathcal{S}_\mu} \sum_{u\in X} (-y)^{-l(vw)-l(u)} x_{\mathbf{ij}} \wr \beta(vw)\beta(u) \\
&= \sum_{v\in Y} \sum_{w\in\mathcal{S}_\lambda\cap\mathcal{S}_\mu} \sum_{u\in X} (-y)^{-l(v)-l(w)-l(u)} x_{\mathbf{ij}} \wr \beta(v)\beta(w)\beta(u) \\
&= \sum_{v\in Y} \sum_{w\in\mathcal{S}_\lambda\cap\mathcal{S}_\mu} \sum_{u\in X} (-y)^{-l(v)-l(wu)} x_{\mathbf{ij}} \wr \beta(v)\beta(wu) \\
&= \sum_{v\in Y} \sum_{w\in\mathcal{S}_\lambda\cap\mathcal{S}_\mu} \sum_{u\in X} (-y)^{-l(v)-l(wu)} \beta(v)\beta(wu) \wr x_{\mathbf{ij}} \\
&= \sum_{v\in Y} \sum_{w''\in\mathcal{S}_\mu} (-y)^{-l(v)-l(w'')} \beta(v)\beta(w'') \wr x_{\mathbf{ij}} \\
&= \sum_{v\in Y} (-y)^{-l(v)} \beta(v) \wr t_q^{\mu}(\mathbf{i}:\mathbf{j}).
\end{aligned}$$

Here, at the fifth step we have used Eq. (5). □

The next result is needed for the transition from t-bideterminants of compositions to T-bideterminants of partitions.

Lemma 9.8. *Let $\lambda \in \Lambda(p,r)$ be a composition and $\mathbf{i},\mathbf{j} \in I(n,r)$. Then the bideterminant $t_q^\lambda(\mathbf{i}:\mathbf{j})$ can be written as a linear combination of bideterminants $T_q^{\lambda'}(\mathbf{k}:\mathbf{l})$.*

Proof. First, there is a permutation $\pi \in \mathcal{S}_p$, such that $\bar\lambda = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(p)}) \in \Lambda^+(p,r)$ is a partition. This $\bar\lambda$ is uniquely determined by λ (but π only under the restriction to be of minimal length). Clearly the parabolic subgroups $\mathcal{S}_\lambda$ and $\mathcal{S}_{\bar\lambda}$ in $\mathcal{S}_r$ are conjugate to each other. Thus, there is an element $v \in \mathcal{S}_r$ such that $v\mathcal{S}_\lambda = \mathcal{S}_{\bar\lambda}v$. Furthermore, it is known from the theory of parabolic subgroups that in the left coset $v\mathcal{S}_\lambda$ and in the right coset $\mathcal{S}_{\bar\lambda}v$ there are unique representatives w (respectively $\bar w$) of minimal length called distinguished coset representatives and that we have $l(wu) = l(w) + l(u)$ for all $u \in \mathcal{S}_\lambda$ and $l(u\bar w) = l(u) + l(\bar w)$ for all $u \in \mathcal{S}_{\bar\lambda}$. Consequently, we have $\beta(w)\kappa_\lambda = \kappa_{\bar\lambda}\beta(\bar w)$. By the definition of bideterminants ($t_q^\lambda(\mathbf{i}:\mathbf{j}) := \kappa_\lambda \wr x_{\mathbf{ij}}$), the relations (5) holding inside $A^{\mathrm{s}}_{R,q}(n,r)$ and the calculus for the symbol $\wr$ given in (3) we obtain

$$\beta(w) \wr t_q^\lambda(\mathbf{i}:\mathbf{j}) = t_q^{\bar\lambda}(\mathbf{i}:\mathbf{j}) \wr \beta(\bar w) = T_q^{\bar\lambda'}(\mathbf{i}:\mathbf{j}) \wr \beta(\bar w).$$

Since $\bar\lambda' = \lambda'$ this results in

$$t_q^\lambda(\mathbf{i}:\mathbf{j}) = \beta(w)^{-1} \wr T_q^{\lambda'}(\mathbf{i}:\mathbf{j}) \wr \beta(\bar w) = \sum_{\mathbf{k},\mathbf{l} \in I(n,r)} \beta(w)^{-1}_{\mathbf{ik}} T_q^{\lambda'}(\mathbf{k}:\mathbf{l}) \beta(\bar w)_{\mathbf{lj}}. \qquad \square$$

Next, we introduce a calculus for our bideterminants bringing our special order $\lhd$ on $I(n,r)$ into the picture. First, some new notation has to be explained. The sum of two multi-indices $\mathbf{i} \in I(n,r)$ and $\mathbf{j} \in I(n,s)$ is defined by juxtaposition, that is

$$\mathbf{i} + \mathbf{j} := (i_1, \dots, i_r, j_1, \dots, j_s) \in I(n, r+s).$$

Note that the map $f : I(n,r) \to \mathbb{N}_0^m$ occurring in Proposition 8.4 is additive in the sense $f(\mathbf{i}+\mathbf{j}) = f(\mathbf{i}) + f(\mathbf{j})$. This implies

$$f(\mathbf{i}+\mathbf{j}) < f(\mathbf{i}+\mathbf{k}) \quad \text{and} \quad f(\mathbf{j}+\mathbf{i}) < f(\mathbf{k}+\mathbf{i}) \quad \text{if} \quad f(\mathbf{j}) < f(\mathbf{k}) \tag{11}$$

with respect to the lexicographic order $<$ on $\mathbb{N}_0^m$. To a multi-index $\mathbf{i} \in I(n,r)$ we consider the following R-spans in $V^{\otimes r}$:

$$W_{\mathbf{i}} := \langle v_{\mathbf{j}} \mid \mathbf{j} \in I(n,r),\ f(\mathbf{j}) < f(\mathbf{i}) \rangle \quad \text{and} \quad \overline{W}_{\mathbf{i}} := \langle v_{\mathbf{j}} \mid \mathbf{j} \in I(n,r),\ f(\mathbf{j}) \leqslant f(\mathbf{i}) \rangle.$$

Furthermore, we set

$$h_{ij} := \begin{cases} q^{-1}, & \text{if } j \neq i, i', \\ 1, & \text{if } j = i \text{ or } j = i' \text{ where } i' > m, \\ q^{-2}, & \text{if } j = i' \leqslant m, \end{cases}$$

and denote the simple transpositions by $s_l = (l, l+1)$, as before. The following lemma is the key concerning calculations with bideterminants. Again we set $y = q^2$.

Lemma 9.9. *For all* $\mathbf{i} \in I(n,r)$ *and* $l \in \underline{r}$ *the following formulas hold in* $V^{\otimes r}$ *modulo the* R*-module* $W_{\mathbf{i}}$:

$$\beta_l(v_{\mathbf{i}}) \equiv \begin{cases} y h_{i_{l+1} i_l} v_{\mathbf{i}s_l} + (y-1)(\mathrm{id}_{V^{\otimes r}} - \gamma_l)(v_{\mathbf{i}}), & \text{if } i_l > i_{l+1}, \\ y h_{i_{l+1} i_l} v_{\mathbf{i}s_l}, & \text{if } i_l \leqslant i_{l+1}, \end{cases}$$

$$\beta_l^{-1}(v_{\mathbf{i}}) \equiv \begin{cases} h_{i_{l+1} i_l} v_{\mathbf{i}s_l} + (y^{-1}-1)(\mathrm{id}_{V^{\otimes r}} - \gamma_l)(v_{\mathbf{i}}), & \text{if } i_l \leqslant i_{l+1}, \\ h_{i_{l+1} i_l} v_{\mathbf{i}s_l}, & \text{if } i_l > i_{l+1}. \end{cases}$$

Proof. The congruence relation for β_l^{-1} follows from the one for β_l because $y\beta^{-1} = \beta + (y-1)(\gamma - \mathrm{id}_{V^{\otimes 2}})$ by (1). Therefore, it is enough to prove the first assertion.

First, consider the case $i_l > i_{l+1}$. If $i_l \neq i'_{l+1}$, the asserted congruence relation is also an equation, as can be seen directly from the definition of β. Turning to the case $i_l = i'_{l+1} =: j \leqslant m$, we split $\mathbf{i}$ into three summands

$$\mathbf{i}^1 = (i_1, \ldots, i_{l-1}), \quad \mathbf{i}^2 = (j', j), \quad \mathbf{i}^3 = (i_{l+1}, \ldots, i_r).$$

To $k \in \underline{n}$ we set $\mathbf{i}(k) := \mathbf{i}^1 + (k, k') + \mathbf{i}^3$ and calculate

$$\beta_l(v_{\mathbf{i}}) = v_{\mathbf{i}s_l} + (y-1)v_{\mathbf{i}} - (y-1)\sum_{k>j} q^{\rho_k - \rho_j} \epsilon_k v_{\mathbf{i}(k)}.$$

Since $(y-1)\sum_{k=1}^{n} q^{\rho_k - \rho_j} \epsilon_k \epsilon_j v_{\mathbf{i}(k)} = (y-1)\gamma_l(v_{\mathbf{i}})$ we obtain the equation

$$\beta_l(v_{\mathbf{i}}) = v_{\mathbf{i}s_l} + (y-1)(\mathrm{id}_{V^{\otimes r}} - \gamma_l)(v_{\mathbf{i}}) + (y-1)\sum_{k\leqslant j} q^{\rho_k - \rho_j} v_{\mathbf{i}(k)}.$$

But $\mathbf{i}(j) = \mathbf{i}s_l$ and

$$f\big(\mathbf{i}(k)\big) = f\big(\mathbf{i}^1\big) + f\big((k,k')\big) + f\big(\mathbf{i}^3\big) < f\big(\mathbf{i}^1\big) + f\big((j',j)\big) + f\big(\mathbf{i}^3\big) = f\big(\mathbf{i}(j')\big) = f(\mathbf{i})$$

for all $k < j$ by (11), yielding the asserted congruence modulo $W_{\mathbf{i}}$. If $i_l < i_{l+1}$ the interesting case is $i'_{l+1} = i_l =: j \leqslant m$. Here the assertion immediately follows from the calculation

$$\beta_l(v_{\mathbf{i}}) = v_{\mathbf{i}s_l} - (y-1)\sum_{k>j'} q^{\rho_k + \rho_j} v_{\mathbf{i}(k)},$$

because $f(\mathbf{i}(k)) < f(\mathbf{i})$ for all $k > j'$. □

Remark 9.10. By Lemma 9.5 the above lemma implies that $\overline{W}_{\mathbf{i}} + \mathcal{J}_r$ is invariant under $\mathcal{A}_r$. But, $W_{\mathbf{i}} = \sum_{f(\mathbf{j})<f(\mathbf{i})} \overline{W}_{\mathbf{j}}$ and thus, $W_{\mathbf{i}} + \mathcal{J}_r$ must be invariant as well.

Corollary 9.11. *Let $\mathbf{j} \in I(n,r)$ and $l \in \underline{r}$. Then to each $\mathbf{k} \in I(n,r)$ satisfying $f(\mathbf{k}) < f(\mathbf{j})$ there is $a_{\mathbf{jk}}(s_l)$ in R (possibly zero) such that the following equations hold in $\mathcal{K}$ for all $\mathbf{i} \in I(n,r)$:*

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \beta_l^{-1} = h_{j_{l+1} j_l} T_q^\lambda(\mathbf{i}:\mathbf{j}s_l) + \sum_{f(\mathbf{k})<f(\mathbf{j})} a_{\mathbf{jk}}(s_l) T_q^\lambda(\mathbf{i}:\mathbf{k}) \quad \text{if } j_l > j_{l+1},$$

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \beta_l = y h_{j_{l+1} j_l} T_q^\lambda(\mathbf{i}:\mathbf{j}s_l) + \sum_{f(\mathbf{k})<f(\mathbf{j})} a_{\mathbf{jk}}(s_l) T_q^\lambda(\mathbf{i}:\mathbf{k}) \quad \text{if } j_l \leqslant j_{l+1}.$$

Proof. Note that $(\beta_l^{-1})_{\mathbf{kj}}$ is the coefficient of $v_{\mathbf{k}}$ in the expression $\beta_l^{-1}(v_{\mathbf{j}})$. By definition of bideterminants and the conventions (10) about $\wr$, the result follows immediately from the lemma. □

Corollary 9.12. *Let $\mathbf{j} \in I(n,r)$ and $w \in \mathcal{S}_{\lambda'}$. Then there is an invertible element $a_{\mathbf{j}}(w) \in R$ and to each $\mathbf{k} \in I(n,r)$ satisfying $f(\mathbf{k}) < f(\mathbf{j})$ another element $a_{\mathbf{jk}}(w)$ in R such that the following equations hold in $\mathcal{K}$ for all $\mathbf{i} \in I(n,r)$:*

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) = a_{\mathbf{j}}(w) T_q^\lambda(\mathbf{i}:\mathbf{j}w) + \sum_{f(\mathbf{k})<f(\mathbf{j})} a_{\mathbf{jk}}(w) T_q^\lambda(\mathbf{i}:\mathbf{k}).$$

Proof. We use induction on the length of w. If this is zero there is nothing to prove. If not, we write $w = w's_l$ where $w', s_l \in \mathcal{S}_{\lambda'}$ and $l(w') = l(w) - 1$. By the induction hypothesis we have

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) = a_{\mathbf{j}}(w') T_q^\lambda(\mathbf{i}:\mathbf{j}w') + \sum_{f(\mathbf{k})<f(\mathbf{j})} a_{\mathbf{jk}}(w') T_q^\lambda(\mathbf{i}:\mathbf{k}).$$

But by Lemma 9.4 we have $T_q^\lambda(\mathbf{i}:\mathbf{j}w') = -T_q^\lambda(\mathbf{i}:\mathbf{j}w') \wr \beta_l^{-1}$ as well as $T_q^\lambda(\mathbf{i}:\mathbf{j}w') = -T_q^\lambda(\mathbf{i}:\mathbf{j}w') \wr \beta_l$. Thus, the assertion follows from the preceding corollary and the fact that $f(\mathbf{j}) = f(\mathbf{j}w')$. □

Lemma 9.13. *Let $\mathbf{i} \in I(n,r)$ satisfy $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$ and let $w \in \mathcal{S}_r$ be arbitrary. Then the following congruence relation holds in $V^{\otimes r}$ modulo the R-submodule $W'_{\mathbf{i}} = W_{\mathbf{i}} + \mathcal{J}_r$:*

$$\beta\big(w^{-1}\big)(v_{\mathbf{i}}) \equiv y^{l(w)} h_{\mathbf{i}}(w) v_{\mathbf{i}w}.$$

Here we have set $h_{\mathbf{i}}(w) := \prod h_{i_{w(k)} i_{w(j)}}$, where the product runs over all pairs $1 \leqslant j < k \leqslant r$ such that $w(j) > w(k)$.

Proof. We use induction on r, the case $r = 1$ being trivial. For $r > 1$ we embed $\mathcal{S}_{r-1}$ as the parabolic subgroup of $\mathcal{S}_r$ generated by $s_1, \ldots, s_{r-2}$, which fix r. If $w \in \mathcal{S}_{r-1}$, there is nothing to prove by the induction hypothesis. Otherwise, we write $w = w's$ where $w' \in \mathcal{S}_{r-1}$ and $s := s_{r-1}s_{r-2}\ldots s_{j+1}s_j$ for an appropriate $j < r$, thus $l(w) = l(w') + r - j$. By the induction hypothesis, Lemma 9.9 and Remark 9.10 we calculate

$$\begin{aligned}\beta\big(w^{-1}\big)(v_{\mathbf{i}}) &\equiv \beta\big(s^{-1}\big)\big(y^{l(w')}h_{\mathbf{i}}(w')v_{\mathbf{i}w'}\big)\\ &= y^{l(w')}h_{\mathbf{i}}(w')\beta_j\beta_{j+1}\dots\beta_{r-1}(v_{\mathbf{i}w'})\\ &\equiv y^{l(w')}y^{r-j}h_{\mathbf{i}}(w')h_{i_{w'(r)}i_{w'(r-1)}}h_{i_{w'(r)}i_{w'(r-2)}}\dots h_{i_{w'(r)}i_{w'(j)}}v_{\mathbf{i}w's}\\ &= y^{l(w)}h_{\mathbf{i}}(w)v_{\mathbf{i}w}. \qquad \Box\end{aligned}$$

Corollary 9.14. *Let $\mathbf{j} \in I(n,r)$ satisfy $j_l \leqslant j_{l+1} \leqslant \dots \leqslant j_{k-1} \leqslant j_k$ for some $1 \leqslant l < k \leqslant r$ and $w \in \mathcal{S}_r$ satisfy $w(i) = i$ for $1 \leqslant i \leqslant l$ or $k < i \leqslant r$. Then, to each $\mathbf{k} \in I(n,r)$ satisfying $f(\mathbf{k}) < f(\mathbf{j})$ there is an element $a'_{\mathbf{jk}}(w)$ in R such that the following equations hold in $\mathcal{K}$ for all $\mathbf{i} \in I(n,r)$:*

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) \wr \beta(w) = y^{l(w)}h_{\mathbf{j}}\big(w^{-1}\big)T_q^\lambda\big(\mathbf{i}:\mathbf{j}w^{-1}\big) + \sum_{f(\mathbf{k})<f(\mathbf{j})} a'_{\mathbf{jk}}(w)T_q^\lambda(\mathbf{i}:\mathbf{k}).$$

Proof. As for the proof of Corollary 9.11, this follows easily from the preceding lemma, Lemma 9.6 and the definition of bideterminants. □

10. The weak straightening algorithm

We are now able to give the proof of the following weak form of the straightening algorithm.

Proposition 10.1 *(Weak quantum symplectic straightening algorithm). Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n,r) \setminus I_\lambda$. Then to each $\mathbf{k} \in I(n,r)$ satisfying $\mathbf{k} \lhd \mathbf{j}$ there is an element $a_{\mathbf{jk}} \in R$ such that in $\mathcal{K}$ the following congruence relation holds for all $\mathbf{i} \in I(n,r)$:*

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k}\lhd\mathbf{j}} a_{\mathbf{jk}}T_q^\lambda(\mathbf{i}:\mathbf{k}) \mod \mathcal{K}(>\lambda).$$

Proof. We divide into the following two cases:

1. $\mathbf{j}$ is not λ-column standard.
2. $\mathbf{j}$ is λ-column standard but not λ-row standard.

Case 1. By assumption there are two consecutive indices j_l and j_{l+1} in $\mathbf{j} = (j_1,\dots,j_r)$ such that $j_l \succeq j_{l+1}$ and $s_l = (l, l+1) \in \mathcal{S}_{\lambda'}$. If $j_l = j_{l+1}$, we have $T_q^\lambda(\mathbf{i}:\mathbf{j}) = 0$ by Corollary 9.2, implying our assertion. In the case $j_{l+1} \succ j_l$ we apply Corollary 9.12:

$$T_q^\lambda(\mathbf{i}:\mathbf{j}) = a_{\mathbf{j}}(s_l)T_q^\lambda(\mathbf{i}:\mathbf{j}s_l) + \sum a_{\mathbf{jk}}(s_l)T_q^\lambda(\mathbf{i}:\mathbf{k}).$$

The multi-indices $\mathbf{k}$ in the sum satisfy $f(\mathbf{k}) < f(\mathbf{j})$ and consequently $\mathbf{k} \lhd \mathbf{j}$. Finally, since $f(\mathbf{j}) = f(\mathbf{j}s_l)$ and $\mathbf{j}s_l$ occurs before $\mathbf{j}$ in the lexicographic order on $I(n,r)$ we have $\mathbf{j}s_l \lhd \mathbf{j}$ as well.

Case 2. In principle we follow the lines of the proof of [Ma, 2.5.7], but since $A^s_{R,q}(n)$ is not commutative we have to work with a fixed basic tableau. The change of basic tableaux in [Ma, 2.5.7] can be compensated for by Lemma 9.8.

To start, let $l \in \underline{r}$ be the smallest index such that j_l is larger than its right-hand neighbour $j_{l'}$ in the λ-tableau of $\mathbf{j}$. Assume that the entry j_l lies in the sth column $\mathbf{j}^s_\lambda$ and that $j_{l'}$ lies in the $(s+1)$th column $\mathbf{j}^{s+1}_\lambda$, where $1 \leqslant s < \lambda_1$. Clearly, $l' = l + \lambda'_s$. Let t be the index of the row containing both entries. We picture this by

$$T^\lambda_{\mathbf{j}} = \cdots \begin{array}{|c|c|c} \multicolumn{1}{c}{\mathbf{j}^s_\lambda} & \multicolumn{1}{c}{\mathbf{j}^{s+1}_\lambda} & \\ \vdots & \vdots & \\ j_{l-1} & j_{l'-1} & t-1 \\ \cline{1-1} j_l & j_{l'} & t \\ \cline{2-2} j_{l+1} & j_{l'+1} & t+1 \\ \vdots & \vdots & \end{array} \cdots.$$

By assumption we have $\cdots \prec j_{l'-1} \prec j_{l'} \prec j_l \prec j_{l+1} \prec \cdots$. Now, we refine the dual partition λ' of λ to a composition $\eta \in \Lambda(p+2, r)$, where $p := \lambda_1$ is the number of columns of the diagram of λ. More precisely, we split the sth and $(s+1)$th column in front of and below the tth row:

$$\eta_i := \begin{cases} \lambda'_i, & i < s, \\ t-1, & i = s, \\ \lambda'_s - t + 1, & i = s+1, \\ t, & i = s+2, \\ \lambda'_{s+1} - t, & i = s+3, \\ \lambda'_{i-2}, & i > s+3, \end{cases} \qquad \mu_i := \begin{cases} \eta_i, & i \leqslant s, \\ \eta_{s+1} + \eta_{s+2}, & i = s+1, \\ \eta_{i+1}, & i > s+2. \end{cases}$$

Obviously, this η is the coarsest refinement of the partition λ' and the composition $\mu \in \Lambda(p+1, r)$ defined above. Let us split the multi-index $\mathbf{j}$ according to η as follows:

$$\mathbf{j}^s_\eta = (j_h, \ldots, j_{l-1}), \qquad \mathbf{j}^{s+1}_\eta = (j_l, \ldots, j_{h+k-1}),$$
$$\mathbf{j}^{s+2}_\eta = (j_{h+k}, \ldots, j_{l'}), \qquad \mathbf{j}^{s+3}_\eta = (j_{l'+1}, \ldots, j_{h+k+k'-1}).$$

Here, $h := l - t + 1 = \lambda'_1 + \cdots + \lambda'_{s-1} + 1$ is the index of the first entry of the sth column and $k := \lambda'_s$ (respectively $k' := \lambda'_{s+1}$) are the lengths of both columns in question. We have

$$\mathbf{j}^s_\lambda = \mathbf{j}^s_\eta + \mathbf{j}^{s+1}_\eta, \quad \mathbf{j}^{s+1}_\lambda = \mathbf{j}^{s+2}_\eta + \mathbf{j}^{s+3}_\eta \quad \text{and set} \quad \mathbf{j}^{s+1}_\mu := \mathbf{j}^{s+1}_\eta + \mathbf{j}^{s+2}_\eta.$$

In order to apply Laplace duality (9.7) to the pair (λ, μ) of compositions we have to choose coset representatives of $\mathcal{S}_\eta = \mathcal{S}_{\lambda'} \cap \mathcal{S}_\mu$ in $\mathcal{S}_{\lambda'}$ and $\mathcal{S}_\mu$ carefully. For our set X we choose distinguished right coset representatives of $\mathcal{S}_\eta$ in $\mathcal{S}_\mu \cong \mathcal{S}_{\mu_1} \times \cdots \times \mathcal{S}_{\mu_{p+1}}$ (cf. proof of Lemma 9.8); in fact, one looks for coset representatives of $\mathcal{S}_{\eta_{s+1}} \times \mathcal{S}_{\eta_{s+2}}$ in $\mathcal{S}_{\mu_{s+1}}$. Since the elements of X are distinguished we have $l(wu) = l(w) + l(u)$ for all $u \in X$ and $w \in \mathcal{S}_\eta$

according to the theory of parabolic subgroups. Similarly, one finds a set Y of distinguished left coset representatives of $\mathcal{S}_\eta$ in $\mathcal{S}_{\lambda'}$ satisfying $l(vw) = l(v) + l(w)$ for $w \in \mathcal{S}_\eta$ and $v \in Y$.

We will not apply Laplace duality to the original index pair $\mathbf{i}, \mathbf{j}$, for we must handle the transition from the order $<$ to $\prec$. Instead of $\mathbf{j}$ we rather consider $\mathbf{j}' := \mathbf{j}w$ where $w \in \mathcal{S}_{\mu_{s+1}} \subseteq \mathcal{S}_r$ is chosen in such a way that $j'_l < j'_{l+1} < \cdots < j'_{l'-1} < j'_{l'}$ and $j'_i = j_i$ for $1 \leqslant i < l$ or $l' < i \leqslant r$ (the embedding of $\mathcal{S}_{\mu_{s+1}}$ is understood according to the composition μ). This w exists uniquely since $\mathbf{j}^{s+1}_\mu = (j_l, \ldots, j_{l'})$ contains exactly $\mu_{s+1} = \lambda'_s + 1$ elements by the assumption $j_{h+k} \prec j_{h-k+1} \prec \cdots \prec j_{l'} \prec j_l \prec \cdots \prec j_{h+k-1}$ on $\mathbf{j}$. Now, by Laplace duality we obtain

$$\sum_{u \in X} (-y)^{-l(u)} T^\lambda_q(\mathbf{i} : \mathbf{j}') \wr \beta(u) = \sum_{v \in Y} (-y)^{-l(v)} \beta(v) \wr t^\mu_q(\mathbf{i} : \mathbf{j}'). \tag{12}$$

With help of Lemma 9.8 the right-hand side of this equation can be written as a linear combination of bideterminants $T^{\mu'}_q(\mathbf{k} : \mathbf{l})$. Thus the right-hand side is seen to lie in $\mathcal{K}(> \lambda)$ as soon we have shown that $\mu' > \lambda$. But that follows since the longest column being removed from the diagram of λ to obtain the diagram of μ' has length λ'_s, whereas a column of length $\mu_{s+1} = \lambda'_s + 1$ has to be added to the diagram of μ'. On the left-hand side of (12) we may apply Corollary 9.14 by construction of the multi-index $\mathbf{j}'$:

$$(-y)^{-l(u)} T^\lambda_q(\mathbf{i} : \mathbf{j}') \wr \beta(u) = \operatorname{sign}(u) h_{\mathbf{j}'}\big(u^{-1}\big) T^\lambda_q\big(\mathbf{i} : \mathbf{j}' u^{-1}\big) + \sum (-y)^{-l(u)} a'_{\mathbf{j}'\mathbf{k}}(u) T^\lambda_q(\mathbf{i} : \mathbf{k}),$$

the sum running over all $\mathbf{k}$ satisfying $f(\mathbf{k}) < f(\mathbf{j}') = f(\mathbf{j})$. Now, for all $u \in X$ we have $\tilde{u} := uw^{-1} \in \mathcal{S}_{\mu_{s+1}}$ since w lies in $\mathcal{S}_{\mu_{s+1}}$. Furthermore, there is a unique coset representative $u_0 \in X$ satisfying $\mathcal{S}_\eta u_0 = \mathcal{S}_\eta w$ and this is the only one for which the corresponding $\tilde{u}$ lies in $\mathcal{S}_\eta$. Therefore, in the case $u \neq u_0$ there is an e such that $l \leqslant e < h + k$ and $h + k \leqslant \tilde{u}^{-1}(e) \leqslant l'$. Choose such an e for each $u \in X$. In doing so, we are assigning a transposition $\hat{u} := (l, e)$ to each u that is contained in $\mathcal{S}_\eta$. In the case of u_0 we set $\hat{u}_0 := \tilde{u}_0 \in \mathcal{S}_\eta$. Applying Corollary 9.12 to $\hat{u}$ one calculates

$$T^\lambda_q\big(\mathbf{i} : \mathbf{j}' u^{-1}\big) = T^\lambda_q\big(\mathbf{i} : \mathbf{j}\tilde{u}^{-1}\big) = a_{\mathbf{j}\tilde{u}^{-1}}(\hat{u}) T^\lambda_q\big(\mathbf{i} : \mathbf{j}\tilde{u}^{-1}\hat{u}\big) + \sum a_{(\mathbf{j}\tilde{u}^{-1})\mathbf{k}}(\hat{u}) T^\lambda_q(\mathbf{i} : \mathbf{k}),$$

where the sum runs over all $\mathbf{k}$ satisfying $f(\mathbf{k}) < f(\mathbf{j}\tilde{u}^{-1}) = f(\mathbf{j})$, again. For these $\mathbf{k}$ we set

$$\bar{a}_{\mathbf{jk}} := \sum_{u \in X} (-y)^{-l(u)} a'_{\mathbf{j}'\mathbf{k}}(u) + \operatorname{sign}(u) h_{\mathbf{j}'}\big(u^{-1}\big) a_{(\mathbf{j}\tilde{u}^{-1})\mathbf{k}}(\hat{u}),$$

whereas in the case $f(\mathbf{k}) = f(\mathbf{j})$ we write

$$\bar{a}_{\mathbf{jk}} := \begin{cases} \operatorname{sign}(u) h_{\mathbf{j}'}(u^{-1}) a_{\mathbf{j}\tilde{u}^{-1}}(\hat{u}), & \text{if there exists } u \in X,\ \mathbf{k} = \mathbf{j}\tilde{u}^{-1}\hat{u}, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\bar{a}_{\mathbf{jj}}$ occurs in the latter definition for $u = u_0$. We assert that for $u \neq u_0$, the multi-index $\mathbf{k} := \mathbf{j}\tilde{u}^{-1}\hat{u}$ occurs before $\mathbf{j}$ in the lexicographic order with respect to $\prec$. For by construction of $\tilde{u}$ and $\hat{u}$ we have

$$k_l = j_{\tilde{u}^{-1}\hat{u}(l)} = j_{\tilde{u}^{-1}(e)} \in \{j_{h+k}, j_{h+k+1}, \ldots, j_{l'}\}$$

and consequently $k_l \prec j_l$. But this implies $\mathbf{k} \prec \mathbf{j}$, since $k_i = j_i$ for $i < l$. Thus we obtain

$$-\bar{a}_{\mathbf{jj}} T_q^{\lambda}(\mathbf{i} : \mathbf{j}) \equiv \sum_{\mathbf{k} \lhd \mathbf{j}} \bar{a}_{\mathbf{jk}} T_q^{\lambda}(\mathbf{i} : \mathbf{k}) \quad \mod \mathcal{K}(> \lambda).$$

Since the coefficient $-\bar{a}_{\mathbf{jj}}$ is invertible, the asserted congruence relation holds as well. $\square$

It should be remarked that the proof works with any other order on $\underline{n}$ instead of $\prec$ as well. The proof of the strong part of the algorithm (Proposition 8.4) can be given right now in the (initial) case $r = 2$ and we are going to do this not only because it is very instructive, but also because we will need a basis of $A^{\mathrm{s}}_{R,q}(n, 2)$ in order to proceed to the general case.

If $r = 2$ there are exactly two partitions in $\Lambda^+(m, 2)$ for $m \geqslant 2$, namely $2\omega_1$ and ω_2, where $\omega_1 = (1)$ and $\omega_2 = (1, 1)$ are the fundamental weights (see Section 3). In the first case we have

$$I_{2\omega_1} = I^{\mathrm{mys}}_{2\omega_1},$$

that is, the weak and the strong form of the straightening algorithm coincide. Turning to ω_2 there is exactly one element in $I_{\omega_2} \setminus I^{\mathrm{mys}}_{\omega_2}$, namely $\mathbf{j} = (m, m')$. By Proposition 4.3 we obtain in $\mathcal{K} = A^{\mathrm{sh}}_{R,q}(n, 2)$

$$T_q^{\omega_2}\big(\mathbf{i} : (m, m')\big) = -q^m \sum_{i=1}^{m-1} q^{-i} T_q^{\omega_2}\big(\mathbf{i} : (i, i')\big)$$

yielding Proposition 8.4 in the case $r = 2$ since $(i, i') \lhd (m, m')$ for all $i < m$.

Remark 10.2. If we had used the notion of symplectic standard tableau instead of the reversed version we would have to consider $(1', 1)$ instead of (m, m') in the last step above. This would force us to work with a reversed version of the order chosen on $\mathbb{N}_0^m$ (as in [O2, Section 7]). But this would cause some trouble concerning Lemma 9.9. One way out could be a manipulation of the Yang–Baxter operator β conjugating it by the twofold tensor product of the appropriate permutation on $\underline{n}$. Thus, one has to decide between working with the familiar version of β or following the familiar notion of tableaux.

11. Quantum symplectic exterior algebra

We are going to prepare the proof of Proposition 8.4 for general r. Since we need a q-analogue of [O2, Lemma 8.1], we have to investigate the quantum symplectic exterior algebra. We start with its definition which can be found in many textbooks on quantum groups (for instance, [CP, Chapter 7]). It is defined as the quotient of the tensor algebra $\mathcal{T}(V) = \bigoplus_{r\in\mathbb{N}_0} V^{\otimes r}$ by a certain ideal. We denote it by $\bigwedge_{R,q}(n)$ and write the symbol $\wedge$ for multiplication in this algebra. Setting

$$c_i := q^i v_{i'} \wedge v_i \quad \text{and} \quad d_i := -q^{-i} v_i \wedge v_{i'}$$

for $i \in \underline{m}$, we write down the defining relations holding in $\bigwedge_{R,q}(n)$ according to [Ha2, (5.2)]:

$$v_k \wedge v_l = -q^{-1} v_l \wedge v_k, \tag{13}$$

$$y^{-i} c_i = y^{-1} d_i + \left(y^{-1} - 1\right) \sum_{j=i+1}^{m} d_j, \tag{14}$$

$$y^i d_i = y c_i + (y-1) \sum_{j=i+1}^{m} c_j, \tag{15}$$

$$v_k \wedge v_k = 0, \tag{16}$$

where $i \in \underline{m}$, $k, l \in \underline{n}$, $k > l$ and $k \neq l'$ is assumed. Remember that the q of [Ha2] corresponds to the inverse of our q. The third relation does not occur in [Ha2] and indeed we have

Lemma 11.1. *Relation* (15) *is a consequence of* (13) *and* (14).

Proof. We use induction on $m - i$. The beginning $y^m d_m = y c_m$ follows directly from $y^{-m} c_m = y^{-1} d_m$ by multiplication with y^{m+1}. For $i < m$ we use (14) and the induction hypothesis to see that

$$\begin{aligned} y^{-1} d_i &= y^{-i} c_i - \left(y^{-1} - 1\right) \sum_{j=i+1}^{m} y^{-j} \left(y c_j + (y-1) \sum_{k=j+1}^{m} c_k \right) \\ &= y^{-i} c_i - \left(y^{-1} - 1\right) \sum_{k=i+1}^{m} \left(y^{1-k} + \sum_{j=i+1}^{k-1} y^{-j} (y-1) \right) c_k. \end{aligned}$$

Since $(y-1)\sum_{j=i+1}^{k-1} y^{-j} = y^{-i} - y^{1-k}$ we obtain (15). □

We set

$$v_I := v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r}, \quad \text{if} \quad I := \{i_1, \ldots, i_r\} \quad \text{and} \quad i_1 < i_2 < \cdots < i_r.$$

In contrast to [O2, Section 7] we take the usual order $<$ on $\underline{n}$ here for technical reasons. A subset $I \subseteq \underline{n}$ ordered in that way will be called an *ordered subset* in the sequel.

Proposition 11.2. *The set $B := \{v_I \mid I \subseteq \underline{n}\}$ is an R-basis of $\bigwedge_{R,q}(n)$.*

Proof. The fact that the set is a set of R-linear generators of $\bigwedge_{R,q}(n)$ follows directly from the relations. Linear independence is shown using the Diamond Lemma for Ring Theory (cf. [Ha1, p. 157]). The technical details can be found in Appendix A.1. $\square$

$\bigwedge_{R,q}(n)$ is a graded algebra since the relations are homogeneous of degree two. A basis for the rth homogeneous summand $\bigwedge_{R,q}(n,r)$ is given by the subset B_r of B corresponding to the set $P(n,r)$ of subsets $I \subseteq \underline{n}$ having cardinality $|I| = r$.

Proposition 11.3. *Considered as elements of $V^{\otimes 2}$ the defining relations precisely span the kernel of the endomorphism $\beta - y\,\mathrm{id}$.*

Proof. Denote the span of the relations by U. It is a matter of calculation to show that

$$\beta\bigl(v_k \wedge v_l + q^{-1} v_l \wedge v_k\bigr) = y\bigl(v_k \wedge v_l + q^{-1} v_l \wedge v_k\bigr),$$

$$\beta(v_k \wedge v_k) = y(v_k \wedge v_k),$$

$$\beta\left(y^{-i} c_i - y^{-1} d_i - \bigl(y^{-1} - 1\bigr) \sum_{j=i+1}^{m} d_j\right) = y^{-i+1} c_i - d_i + (y-1) \sum_{j=i+1}^{m} d_j,$$

which is only hard in the last case. The technical details of that calculation can be found in Appendix A.3. From these equations and Lemma 11.1 we see that U is contained in the kernel of $(\beta - y\,\mathrm{id}_{V^{\otimes 2}})$.

To show the other inclusion we consider two free R submodules

$$W^1 := \langle v_i \otimes v_j \mid 1 \leqslant i < j \leqslant n\rangle_R \quad \text{and} \quad W^2 := \langle v_j \otimes v_i \mid 1 \leqslant i \leqslant j \leqslant n\rangle_R$$

of $V^{\otimes 2}$. Now, there are two direct sum decompositions of R-modules $V^{\otimes 2} = W^1 \oplus U$ and $V^{\otimes 2} = W^1 \oplus W^2$. Let v be in the kernel of $(\beta - y\,\mathrm{id}_{V^{\otimes 2}})$. We may write $v = w_1 + u$ where $u \in U$ and $w_1 \in W^1$. From the definition of β it follows that $\beta(W^1) \subseteq W^2$. Consequently, since $\beta(u) = yu$ there is a $w_2 \in W^2$ such that $(\beta - y\,\mathrm{id}_{V^{\otimes 2}})(v) = -yw_1 + w_2$. Thus, $(\beta - y\,\mathrm{id}_{V^{\otimes 2}})(v) = 0$ implies $w_1 = 0$, that is $v \in U$. $\square$

Proposition 11.4. *$\bigwedge_{R,q}(n,2)$ is an $A^{\mathrm{s}}_{R,q}(n,2)$ comodule.*

Proof. By the previous proposition we have to show that the kernel of $\beta - y\,\mathrm{id}$ is an $A := A^{\mathrm{s}}_{R,q}(n,2)$ subcomodule of $V^{\otimes 2}$. Call this kernel U and let $r \in U$. We must show $\tau(r) \in A \otimes U$. By construction of A as universal coalgebra with the property that β and γ are morphisms of the A-comodule $V^{\otimes 2}$ (see [O2, Section 2, definition of $M(A)$] and Eq. (5) ff.) we see

$$y\tau(r) = \tau\bigl(\beta(r)\bigr) = \mathrm{id}_A \otimes \beta\bigl(\tau(r)\bigr).$$

But this means $\mathrm{id}_A \otimes (\beta - y\,\mathrm{id})(\tau(r)) = 0$. Since A, U and $\bigwedge_{R,q}(n,2)$ are free R-modules we may conclude $\tau(r) \in A \otimes U$. $\square$

If B is a bialgebra and A an algebra that is a B-comodule we call A a B-comodule algebra if multiplication as well as the embedding of the unit element are morphisms of comodules. For example the tensor algebra $\mathcal{T}(V) = \bigoplus_{r\in\mathbb{N}_0} V^{\otimes r}$ over R with multiplication given on homogeneous summands by

$$\nabla : V^{\otimes r} \otimes V^{\otimes s} \to V^{\otimes r+s}, \quad \nabla(v_{\mathbf{i}} \otimes v_{\mathbf{j}}) := v_{\mathbf{i}+\mathbf{j}}$$

and embedding

$$\iota : R \to V^{\otimes 0}, \quad \iota(x) := x 1_{\mathcal{T}(V)}$$

is an $A := A^{\mathrm{s}}_{R,q}(n)$-comodule algebra, since $(\nabla \otimes \mathrm{id}_A) \circ (\tau_r \otimes \tau_s) = \tau_{r+s} \circ \nabla$ and $(\iota \otimes \mathrm{id}_A) \circ \tau_R = \tau_0 \otimes \iota$. Here we have written $\tau_r \otimes \tau_s$, τ_{r+s}, τ_R and τ_0 for the comodule structure maps of $V^{\otimes r} \otimes V^{\otimes s}$, $V^{\otimes r+s}$, R and $V^{\otimes 0}$, respectively.

Proposition 11.5. *$\bigwedge_{R,q}(n)$ is an $A^{\mathrm{s}}_{R,q}(n)$-comodule algebra.*

Proof. As pointed out above the tensor algebra $\mathcal{T}(V)$ over R has a natural structure of an $A^{\mathrm{s}}_{R,q}(n)$-comodule algebra. Consequently by multiplicativity and the proof of Proposition 11.4 the ideal generated by the kernel of $\beta - y\,\mathrm{id}$ is an $A^{\mathrm{s}}_{R,q}(n)$-comodule. But this is precisely the defining ideal of $\bigwedge_{R,q}(n)$ by Proposition 11.3. Thus $\bigwedge_{R,q}(n)$ inherits the comodule algebra structure from $\mathcal{T}(V)$. $\square$

Denote the comodule structure map of $\bigwedge_{R,q}(n)$ by $\tau_\wedge : \bigwedge_{R,q}(n) \to \bigwedge_{R,q}(n) \otimes A^{\mathrm{s}}_{R,q}(n)$.

Proposition 11.6. *The coefficient functions of $\bigwedge_{R,q}(n,r)$ are given by*

$$\tau_\wedge(v_J) = \sum_{I\in P(n,r)} v_I \otimes T_q^{\omega_r}(\mathbf{i} : \mathbf{j}),$$

where $\mathbf{i} = (i_1, \dots, i_r)$ and $\mathbf{j} = (j_1, \dots, j_r)$ are the multi-indices corresponding to the ordered subsets $I := \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$, respectively.

Let us first treat the ingredients needed in the proof of Proposition 11.6.

Lemma 11.7. *Let $\pi_r : V^{\otimes r} \to \bigwedge_{R,q}(n,r)$ be the natural projection. Then the endomorphism $\kappa_\lambda := \sum_{w\in\mathcal{S}_\lambda} (-y)^{-l(w)} \beta(w)$ factors through π_r, i.e. there is a homomorphism of R-modules $\nu_r : \bigwedge_{R,q}(n,r) \to V^{\otimes r}$ such that $\kappa_r = \nu_r \circ \pi_r$.*

Proof. Since the defining ideal of $\bigwedge_{R,q}(n)$ is generated by the kernel of $(\mathrm{id}_{V^{\otimes 2}} - y^{-1}\beta)$ by Proposition 11.3 the assertion immediately follows from Lemma 9.1. $\square$

Let $I_{\omega_r}^{<} := \{\mathbf{i} \in I(n,r) \mid i_1 < i_2 < \cdots < i_r\}$ be the set of multi-indices corresponding to the ordered subsets $I \in P(n,r)$.

Lemma 11.8. *Let F_r be the R-linear span of $\{v_{\mathbf{j}} \mid \mathbf{j} \in I(n,r) \setminus I_{\omega_r}^{<}\}$ in $V^{\otimes r}$. Then for all $w \in \mathcal{S}_r \setminus \{\mathrm{id}\}$ and $\mathbf{i} \in I_{\omega_r}^{<}$ it follows that $\beta(w)(v_{\mathbf{i}}) \in F_r$.*

Proof. We use induction on r. The case $r = 2$ directly follows from the formulas

$$\beta(v_{(k,l)}) = q v_{(l,k)}, \tag{17}$$

$$\beta(v_{(i,i')}) = v_{(i',i)} + (y-1)\sum_{j=1}^{i-1} q^{j-i} v_{(j',j)}, \tag{18}$$

which are valid for $k < l$, $k \neq l'$ and $i \leqslant m$. If $r > 2$, we embed $\mathcal{S}_{r-1}$ as the subgroup of $\mathcal{S}_r$ that fixes the letter r. If $w \in \mathcal{S}_{r-1}$, there is nothing to prove since $F_{r-1} \otimes V \subseteq F_r$. Otherwise, we may write $\beta(w) = \beta(w')\beta_{r-1}\beta_{r-2}\dots\beta_l$ where $w' \in \mathcal{S}_{r-1}$ and $l \leqslant r-1$.

First consider the case where i_l' is not contained in $\{i_{l+1}, \dots, i_r\}$. Applying $\beta_{r-1}\beta_{r-2}\dots\beta_l$ to $v_{\mathbf{i}}$ we only have to use (17) but not (18). Consequently, we have

$$\beta_{r-1}\beta_{r-2}\dots\beta_l(v_{\mathbf{i}}) = q^{r-l} v_{i_1}\dots\hat{v}_{i_l}\dots v_{i_r} v_{i_l}.$$

Here, $\hat{v}_{i_l}$ denotes the omission of v_{i_l}. This element obviously lies in F_r, proving the assertion in the case $w' = \mathrm{id}$. If w' is not the identity map we have

$$\beta(w')\bigl(q^{r-l+1} v_{i_1}\dots\hat{v}_{i_l}\dots v_{i_r} v_{i_l}\bigr) \in F_{r-1} \otimes v_{i_l} \subseteq F_r$$

by the induction hypothesis since $(i_1, \dots, \hat{i}_l, \dots, i_r) \in I_{\omega_{r-1}}^{<}$.

We next consider the case $i_l' \in \{i_{l+1}, \dots, i_r\}$. This forces $i_l \leqslant m$ because $i_l < i_l'$. Let $i_l' = i_k$. As above, we have

$$\beta_{k-2}\beta_{k-3}\dots\beta_l(v_{\mathbf{i}}) = q^{k-l-1} v_{i_1}\dots\hat{v}_{i_l}\dots v_{i_{k-1}} v_{i_l} v_{i_k} v_{i_{k+1}}\dots v_{i_r}.$$

Applying β_{k-1} to this expression, we have to use (18) for the first time. But for each basis element $v_{\mathbf{j}}$ occurring as a summand in the resulting expression we have $j_k \leqslant i_l \leqslant m$. Similar things happen concerning the remaining $\beta_k, \dots, \beta_{r-1}$. Thus, for each $v_{\mathbf{j}}$ occurring as a summand in $\beta_{r-1}\beta_{r-2}\dots\beta_l(v_{\mathbf{i}})$, it follows that $j_r \leqslant i_l \leqslant m$. On the other hand, for each such summand there must exist an $h < r$ where $j_h > m$. This is because $\mathbf{j}$ must contain a pair $\{i, i'\}$ for some $i \in \underline{m}$, since this was the case for the multi-index $\mathbf{i}$ we started with and β either exchanges the position of such a pair or replaces it by a sum where other such pairs occur in each summand. Consequently, we obtain $\mathbf{j} \in F_r$ in this case too. $\square$

Let the coefficient matrices of the R-module homomorphisms π_r, κ_r and ν_r (from Lemma 11.7) be given by

$$\pi_r(v_{\mathbf{j}}) = \sum_{I \in P(n,r)} \pi_{I\mathbf{j}} v_I, \quad \kappa_r(v_{\mathbf{j}}) = \sum_{\mathbf{i} \in I(n,r)} \kappa_{\mathbf{ij}} v_{\mathbf{i}} \quad \text{and} \quad \nu_r(v_J) = \sum_{\mathbf{i} \in I(n,r)} \nu_{\mathbf{i}J} v_{\mathbf{i}}.$$

Now, if $\mathbf{j} \in I_{\omega_r}^<$ corresponds to the ordered set $J \in P(n,r)$ we have $\pi_r(v_{\mathbf{j}}) = v_J$ yielding $\nu_{\mathbf{i}J} = \kappa_{\mathbf{ij}}$ by Lemma 11.7. From Lemma 11.8 it follows that $\kappa_r(v_{\mathbf{j}}) \equiv v_{\mathbf{j}}$ modulo F_r. Thus, for a pair $\mathbf{i}, \mathbf{j} \in I_{\omega_r}^<$ of multi-indices corresponding to ordered sets $I, J \in P(n,r)$, we obtain $\nu_{\mathbf{i}J} = \kappa_{\mathbf{ij}} = \delta_{IJ}$ (Kronecker symbol). Finally, from $\kappa_r = \nu_r \circ \pi_r$ we see for all $\mathbf{i} \in I_{\omega_r}^<$ and $\mathbf{j} \in I(n,r)$

$$\kappa_{\mathbf{ij}} = \sum_{K \in P(n,r)} \nu_{\mathbf{i}K} \pi_{K\mathbf{j}} = \pi_{I\mathbf{j}}. \tag{19}$$

We are now ready to give the proof of Proposition 11.6. We calculate

$$\begin{aligned} \tau_\wedge(v_J) &= \sum_{\mathbf{k} \in I(n,r)} v_{k_1} \wedge \cdots \wedge v_{k_r} \otimes x_{\mathbf{kj}} = \sum_{\mathbf{k} \in I(n,r)} \sum_{I \in P(n,r)} \pi_{I\mathbf{k}} v_I \otimes x_{\mathbf{kj}} \\ &= \sum_{I \in P(n,r)} v_I \otimes \sum_{\mathbf{k} \in I(n,r)} \kappa_{\mathbf{ik}} x_{\mathbf{kj}}. \end{aligned}$$

But, this is exactly what we wanted by the definition $T_q^{\omega_r}(\mathbf{i} : \mathbf{j}) = \kappa_r \wr x_{\mathbf{ij}}$ of bideterminants.

The formula we just have proved has some useful consequences concerning the comultiplication and augmentation of $A := A_{R,q}^{s}(n)$. These are valid for any pair $\mathbf{i}, \mathbf{j} \in I_{\omega_r}^<$ of multi-indices corresponding to ordered sets $I, J \in P(n,r)$ and follow directly with the help of the comodule axioms $(\tau_\wedge \otimes \mathrm{id}_A) \circ \tau_\wedge = (\mathrm{id}_\wedge \otimes \Delta) \circ \tau_\wedge$ and $(\mathrm{id}_\wedge \otimes \epsilon) \circ \tau_\wedge = \mathrm{id}_\wedge$:

$$\Delta\big(T_q^{\omega_r}(\mathbf{i} : \mathbf{j})\big) = \sum_{\mathbf{k} \in I_{\omega_r}^<} T_q^{\omega_r}(\mathbf{i} : \mathbf{k}) \otimes T_q^{\omega_r}(\mathbf{k} : \mathbf{j}), \tag{20}$$

$$\epsilon\big(T_q^{\omega_r}(\mathbf{i} : \mathbf{j})\big) = \delta_{\mathbf{ij}}. \tag{21}$$

Another useful consequence is the following corollary.

Corollary 11.9. *Let $a_{\mathbf{j}} \in R$ be such that $\sum_{\mathbf{j} \in I(n,r)} a_{\mathbf{j}} v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_r} = 0$. Then for all $\mathbf{i} \in I(n,r)$ we have*

$$\sum_{\mathbf{j} \in I(n,r)} a_{\mathbf{j}} T_q^{\omega_r}(\mathbf{i} : \mathbf{j}) = \sum_{\mathbf{j} \in I(n,r)} a_{\mathbf{j}} T_q^{\omega_r}(\mathbf{j} : \mathbf{i}) = 0.$$

Proof. By Lemma 11.7 and the assumption we have

$$\kappa_r\left(\sum_{\mathbf{j} \in I(n,r)} a_{\mathbf{j}} v_{\mathbf{j}}\right) = \sum_{\mathbf{j} \in I(n,r)} a_{\mathbf{j}} \kappa_r(v_{\mathbf{j}}) = 0.$$

Consequently, for all $\mathbf{k} \in I(n,r)$ we obtain $\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}}\kappa_{\mathbf{kj}} = 0$ and therefore

$$\sum_{\mathbf{j}\in I(n,r)} a_{\mathbf{j}} T_q^{\omega_r}(\mathbf{i}:\mathbf{j}) = \sum_{\mathbf{k},\mathbf{j}\in I(n,r)} a_{\mathbf{j}} x_{\mathbf{ik}}\kappa_{\mathbf{kj}} = 0.$$

The equation with exchanged indices is deduced by an application of the involution $*$ according to (8). □

12. Proof of Proposition 8.4

First, we have to state the q-analogue of [O2, Lemma 8.1], one of the principal ingredients in the proof of the symplectic straightening algorithm in the classical case. In order to define the quantum analogue to the ideal N considered there we have to look in more detail at the elements c_i and d_i defined in the previous section.

Relation (16) implies $c_i \wedge d_i = d_i \wedge c_i = 0$. Consequently we get from (14) and (15)

$$d_i^2 := d_i \wedge d_i = (y-1)\sum_{j=i+1}^{m} d_i \wedge d_j \quad \text{and} \quad c_i^2 = \left(y^{-1}-1\right)\sum_{j=i+1}^{m} c_i \wedge c_j. \tag{22}$$

This stands in remarkable contrast to the classical and even quantum linear case where such expressions vanish. On the other hand, by (13) and the above explanations all of the elements c_i and d_j commute pairwise with each other, exactly as in the classical case. Consequently, they generate a commutative subalgebra of $\bigwedge_{R,q}(n)$ and the elements $d_K := d_{k_1} \wedge d_{k_2} \wedge \cdots \wedge d_{k_a}$ are defined independently of the order of the elements of the subset $K := \{k_1,\ldots,k_a\} \subseteq \underline{m}$. Again, we write $P(m,a)$ for the collection of all subsets K of $\underline{m}$ whose cardinality is a. Set

$$D_a := \sum_{K\in P(m,a)} d_K$$

and let N be the ideal in $\bigwedge_{R,q}(n)$ generated by the elements $D_1, D_2, \ldots, D_m$. We call an ordered subset $I \in P(n,r)$ *reverse symplectic* if the multi-index $\mathbf{i}w$ obtained from I by ordering its elements according to $\prec$ (obtained from $\mathbf{i}$ by a suitable permutation $w \in \mathcal{S}_r$ such that $i_{w(1)} \prec i_{w(2)} \prec \cdots \prec i_{w(r)}$) is ω_r-reverse symplectic standard. Here ω_r is the rth fundamental weight.

Proposition 12.1. *Let $I \in P(n,r)$ be non-reverse symplectic. Then, to each $J \in P(n,r)$ such that the inequality $f(\mathbf{j}) < f(\mathbf{i})$ holds for the corresponding multi-indices $\mathbf{i}$ and $\mathbf{j}$, there exists $a_{IJ} \in R$ such that in $\bigwedge_{R,q}(n)$ the following congruence relation holds*:

$$v_I \equiv \sum_{J\in P(n,r),\ f(\mathbf{j})<f(\mathbf{i})} a_{IJ} v_J \mod N.$$

Proposition 12.2. *The semibialgebra (see Definition* 8.1) $A^{\mathrm{sh}}_{R,q}(n)$ *acts trivially on the elements* D_a, *that is* $\tau_\wedge(D_a) = 0$.

We postpone the very technical proofs of both propositions to separate sections below.

Let us prove Proposition 8.4 in the case $\lambda = \omega_r$ first. Take $\mathbf{j} \in I(n,r) \setminus I^{\mathrm{mys}}_{\omega_r}$. Using the weak part of the straightening algorithm (10.1), we may assume $\mathbf{j} \in I_{\omega_r} \setminus I^{\mathrm{mys}}_{\omega_r}$. This means $j_1 \prec j_2 \prec \cdots \prec j_r$. In order to apply our lemmas we have to change orders from $\prec$ to $<$. Let $w \in \mathcal{S}_r$ be such that $j_{w(1)} < j_{w(2)} < \cdots < j_{w(r)}$, that is, $\mathbf{j}w$ is a multi-index corresponding to a non-reverse symplectic ordered set $J \in P(n,r)$ in the sense of Proposition 12.1. Application of this proposition to v_J yields

$$X := v_J - \sum_{K \in P(n,r),\, f(\mathbf{k}) < f(\mathbf{j})} a_{\mathbf{jk}} v_K \in N$$

since $f(\mathbf{j}) = f(\mathbf{j}w)$. According to Proposition 12.2, the element $\tau_\wedge(X)$ must be zero. Applying Proposition 11.6, we obtain the following equation holding in $\bigwedge_{R,q}(n,r) \otimes \mathcal{K}$:

$$\sum_{I \in P(n,r)} v_I \otimes \left(T_q^{\omega_r}(\mathbf{i} : \mathbf{j}w) - \sum_{K \in P(n,r),\, \mathbf{k} \lhd \mathbf{j}} a_{\mathbf{jk}} T_q^{\omega_r}(\mathbf{i} : \mathbf{k}) \right) = 0.$$

Since $\{v_I \mid I \in P(n,r)\}$ is a basis of $\bigwedge_{R,q}(n,r)$, each individual summand in the summation over $P(n,r)$ must be zero. Together with Corollary 9.12, this gives the desired result in the case of multi-indices $\mathbf{i}$ corresponding to ordered subsets $I \in P(n,r)$, that is $\mathbf{i} \in I^{<}_{\omega_r}$. The case for general $\mathbf{i}$ can be deduced from this using

$$T_q^{\omega_r}(\mathbf{i} : \mathbf{j}) = \sum_{K \in P(n,r)} \nu_{\mathbf{i}K} T_q^{\omega_r}(\mathbf{k} : \mathbf{j}),$$

which follows from the formula $\kappa_r = \nu_r \circ \pi_r$ of Lemma 11.7 together with (19).

Next we consider the general case of λ. Here, we can proceed exactly as in the classical case. Again, we may assume $\mathbf{j} \in I_\lambda \setminus I^{\mathrm{mys}}_\lambda$ by the weak part of the straightening algorithm. Let $\lambda' = (\mu_1, \ldots, \mu_p)$ be the dual partition ($p = \lambda_1$). We split $\mathbf{j}$ into p multi-indices $\mathbf{j}^l \in I(n, \mu_l)$, where for each $l \in \underline{p}$ the entries of $\mathbf{j}^l$ are taken from the lth column of $T^\lambda_{\mathbf{j}}$. The same thing can be done with $\mathbf{i}$. Since $\mathbf{j}$ is not λ-reverse symplectic standard but standard there must be a column s such that $\mathbf{j}^s$ is not ω_{μ_s}-reverse symplectic standard. Applying the result to the known case of $T_q^{\omega_{\mu_s}}(\mathbf{i}^s : \mathbf{j}^s)$, we obtain

$$\begin{aligned} T_q^\lambda(\mathbf{i} : \mathbf{j}) &= T_q^{\omega_{\mu_1}}\big(\mathbf{i}^1 : \mathbf{j}^1\big) T_q^{\omega_{\mu_2}}\big(\mathbf{i}^2 : \mathbf{j}^2\big) \cdots T_q^{\omega_{\mu_s}}\big(\mathbf{i}^s : \mathbf{j}^s\big) \cdots T_q^{\omega_{\mu_p}}\big(\mathbf{i}^p : \mathbf{j}^p\big) \\ &\equiv \sum a_{\mathbf{j}^s\mathbf{k}^s} T_q^{\omega_{\mu_1}}\big(\mathbf{i}^1 : \mathbf{j}^1\big) \cdots T_q^{\omega_{\mu_s}}\big(\mathbf{i}^s : \mathbf{k}^s\big) \cdots T_q^{\omega_{\mu_p}}\big(\mathbf{i}^p : \mathbf{j}^p\big) = \sum a_{\mathbf{jk}} T_q^\lambda(\mathbf{i} : \mathbf{k}). \end{aligned}$$

The element $\mathbf{k}^s \in I(n, \mu_s)$ satisfies $\mathbf{k}^s \lhd \mathbf{j}^s$, $\mathbf{k} \in I(n,r)$ is constructed from $\mathbf{j}$ by replacing the entries of $\mathbf{j}^s$ by that of $\mathbf{k}^s$ and $a_{\mathbf{jk}}$ is the same as $a_{\mathbf{j}^s\mathbf{k}^s}$ for the corresponding $\mathbf{k}^s$. The product formula for bideterminants applied above is valid by our choice of basic

λ-tableaux inserting the numbers $1, \ldots, r$ column by column top down (otherwise the non-commutativity of $A^{\mathrm{s}}_{R,q}(n)$ would cause some trouble). From (11) we see $\mathbf{k} \lhd \mathbf{j}$ and the proof of 8.4 is completed.

13. Proof of Proposition 12.1

For convenience, in the following sections we abbreviate $y = q^2$ and denote multiplication in $\bigwedge_{R,q}(n)$ by juxtaposition instead of $\wedge$. Furthermore, to sets $K, L \subseteq \underline{m}$ we associate integers

$$v(K, L) := \left|\left\{(k, l) \in K \times L \mid k > l\right\}\right|.$$

Lemma 13.1. *Let $a \in \underline{m}$. If $\underline{m} = L \cup M$ is a partition of $\underline{m}$ into disjoint subsets L and M then to each $K \in P(m, a)$ there is an integer $s(K, L)$ such that*

$$D_a = \sum_{K \in P(m,a)} y^{s(K,L)} c_{K \cap L} d_{K \cap M}.$$

If $K \subseteq M$, the integer $s(K, L)$ equals $v(K, L)$.

Sketch of proof. In order to prove this, one has to use a more general statement in which the set $\underline{m}$ is substituted by $\{l, l+1, \ldots, m\}$ for some $l \in \underline{m}$. After this, the results can be proved straightforwardly using induction on $m - l$ and the relations (14) and (15) of the exterior algebra. For the details we refer to Appendix A.3. $\square$

To a set $I \in P(n, r)$ we associate the following subsets of $\underline{m}$:

$$I^- := I \cap \underline{m}, \quad I^+ := \{i \in \underline{m} \mid i' \in I\} \quad \text{and} \quad I^0 := I^- \cap I^+.$$

Lemma 13.2. *Let $I \in P(n, r)$ be such that $I^0 = \emptyset$ and $J \subseteq \underline{m}$. Set $s = |J|$ and $T := \underline{m} \setminus (I^- \cup I^+)$. If $J \subseteq T$ and $a \in \underline{m}$ is such that $s < a$ then we have*

$$v_I d_J D_{a-s} = v_I \sum_{S \in P(m,a),\ J \subseteq S \subseteq T} y^{v(S \setminus J, I^+ \cup J)} d_S.$$

Proof. We apply Lemma 13.1 to $L := I^+ \cup J$ and $M := \underline{m} \setminus L$. Since $J \subseteq T$ there is an invertible element $b \in R$ such that $v_I d_J = b d_J v_I$ by (13). We claim that $v_I d_J c_{K \cap L} d_{K \cap M}$ vanishes for each $K \in P(m, a-s)$ that is not contained in $T \setminus J$. If $K \cap J \neq \emptyset$ then $d_J c_{K \cap L} = 0$. If $K \cap I^+ \neq \emptyset$ and $K \cap J = \emptyset$ then $v_I d_J c_{K \cap L} = v_I c_{K \cap L} d_J = 0$. The only other possibility is $K \subseteq M$, $K \cap I^- \neq \emptyset$ and $K \cap J = \emptyset$, in which case $v_I d_J c_{K \cap L} d_{K \cap M} = v_I d_{K \cap M} d_J c_{K \cap L} = 0$. Thus the expression is non-zero only if $K \subseteq T \setminus J$. A set $K \subseteq T \setminus J$ is obviously contained in M, so by the second part of Lemma 13.1 we have $s(K, L) = v(K, L) = v(K, I^+ \cup J)$. Thus, setting $S = J \cup K$ the assertion follows. $\square$

Lemma 13.3. *Let $M \subseteq K \subseteq \underline{m}$ be fixed. Then*

$$\sum_{L \subseteq K \setminus M} (-1)^{|L|} y^{v(K,L)} = \begin{cases} 1, & K = M, \\ 0, & K \neq M. \end{cases}$$

Proof. Clearly the sum is 1 if $K = M$ since $v(K, \emptyset) = 0$. For $K \neq M$ we show by induction on $n := |K| > 0$ that the sum is zero. Starting with the case $n = 1$ we have $M = \emptyset$ and

$$\sum_{L \subseteq K} (-1)^{|L|} y^{v(K,L)} = y^{v(K,\emptyset)} - y^{v(K,K)} = 1 - 1 = 0.$$

For the induction step, let $n > 1, x \in K$ be minimal and set $\widehat{K} := K \setminus x$. If $x \notin M$ we calculate

$$\sum_{L \subseteq K \setminus M} (-1)^{|L|} y^{v(K,L)} = \sum_{L \subseteq \widehat{K} \setminus M} (-1)^{|L|} y^{v(K,L)} + \sum_{L \subseteq \widehat{K} \setminus M} (-1)^{|L|+1} y^{v(K,L \cup \{x\})}.$$

Since x is minimal, $v(K, L) = v(\widehat{K}, L)$ and $v(K, L \cup \{x\}) = v(\widehat{K}, L) + n - 1$ if $L \subseteq \widehat{K} \setminus M$. Now, we may apply the induction hypothesis to $\widehat{K}$ which results in

$$\sum_{L \subseteq K \setminus M} (-1)^{|L|} y^{v(K,L)} = \left(1 - y^{n-1}\right) \sum_{L \subseteq \widehat{K} \setminus M} (-1)^{|L|} y^{v(\widehat{K},L)} = 0.$$

In the case $x \in M$, we write $\widehat{M} := M \setminus \{x\}$. Similarly, we calculate

$$\sum_{L \subseteq K \setminus M} (-1)^{|L|} y^{v(K,L)} = \sum_{L \subseteq \widehat{K} \setminus \widehat{M}} (-1)^{|L|} y^{v(\widehat{K},L)},$$

where the right-hand side is zero by the induction hypothesis. $\square$

Lemma 13.4. *Let $I \in P(n, r)$ with $r > m$ and set $a = |I^0|$, $\hat{I} := I \setminus \{i, i' \mid i \in I^0\}$ and $T := \underline{m} \setminus (I^+ \cup I^-)$. Then there is an invertible $a_I \in R$ such that the following equation holds*:

$$v_{\hat{I}} \sum_{J \subseteq T} (-1)^{|J|} y^{v(J,(\hat{I})^+ \cup J)} d_J D_{a-|J|} = a_I v_I.$$

Proof. Let $\widehat{T} = T \cup I^0 = \underline{m} \setminus ((\hat{I})^+ \cup (\hat{I})^-)$ and observe that $(\hat{I})^+ = I^+ \setminus I^0$, $(\hat{I})^0 = \emptyset$, $(\hat{I})^+ \cap J = \emptyset$ and that $T \cap I^0 = \emptyset$. Because $r > m$, we must have $a \geqslant 1$. On the other hand, $2a + |\hat{I}| = r > m$ implies $a > m - |\hat{I}| - |I^0| = |T| \geqslant |J|$ for a set J as in the sum. Therefore we may apply Lemma 13.2:

$$\begin{aligned}
& v_{\hat{I}} \sum_{J \subseteq T} (-1)^{|J|} y^{v(J,(\hat{I})^+ \cup J)} d_J D_{a-|J|} \\
&\quad = v_{\hat{I}} \sum_{J \subseteq T} (-1)^{|J|} y^{v(J,(\hat{I})^+ \cup J)} \sum_{S \in P(m,a),\, J \subset S \subseteq \widehat{T}} y^{v(S \setminus J,(\hat{I})^+ \cup J)} d_S \\
&\quad = v_{\hat{I}} \sum_{J \subseteq S \subseteq \widehat{T},\, J \cap I^0 = \emptyset} (-1)^{|J|} y^{v(S,J)+v(S,(\hat{I})^+)} d_S \\
&\quad = v_{\hat{I}} \sum_{S \subseteq \widehat{T}} y^{v(S,(\hat{I})^+)} d_S \sum_{J \subseteq S \setminus I^0} (-1)^{|J|} y^{v(S,J)}.
\end{aligned}$$

But by Lemma 13.3, the last term equals $y^{v(I^0,\hat{I}^+)} v_{\hat{I}} d_{I^0}$. Using relation (13) of the exterior algebra, this can be transformed into v_I up to some invertible multiple a_I. $\square$

We are now able to prove Proposition 12.1 by induction on the Lie rank m. In the case where $m = 1$ both sets of $P(2,1) = \{\{1\},\{2\}\}$ are reverse symplectic. In $P(2,2)$ there is just one set, namely $I = \{1,2\}$, for which we have $v_I = -qd_1 = -qD_1 \in N$. Thus there is nothing to prove here.

For the induction step we embed $\bigwedge_{R,q}(n-2)$ into $\bigwedge_{R,q}(n)$ sending v_i to v_{i+1}. It is easy to check that this indeed leads to an embedding of algebras. Using the induction hypothesis we may treat the case where $I \subseteq \underline{n} \setminus \{1,n\}$ without much effort. Some caution is needed only concerning the difference between the two ideals N of $\bigwedge_{R,q}(n-2)$ and $\bigwedge_{R,q}(n)$. Denote them by $N(n-2)$ and $N(n)$. A single element $u \in N(n-2)$ can be written

$$u = \sum_{\{1,n\} \subseteq L \subseteq \underline{n}} a_{\hat{I}L} v_L \mod N(n),$$

where the basis elements v_L all are smaller than v_I, that is $f(\mathbf{l}) < f(\mathbf{i})$ for the corresponding multi-indices. If $I \cap \{1,n\} \neq \emptyset$ we may apply the induction hypothesis to the set $\hat{I} := I \setminus \{1,n\}$ in case $\hat{I}$ is non-reverse symplectic too:

$$v_{\hat{I}} \equiv \sum_{\hat{J} \subseteq \underline{n} \setminus \{1,n\},\, |\hat{I}| = |\hat{J}|,\, f(\hat{\mathbf{j}}) < f(\hat{\mathbf{i}})} a_{\hat{I}\hat{J}} v_{\hat{J}} + \sum_{\{1,n\} \subseteq L \subseteq \underline{n}} a_{\hat{I}L} v_L \mod N(n).$$

Again, the second sum compensate for the difference between the two ideals $N(n-2)$ and $N(n)$. Multiplying this congruence by v_1 from the left (respectively by v_n from the right, respectively by both from both sides) yields the assertion because the elements v_L vanish, and $f(\hat{\mathbf{j}}) < f(\hat{\mathbf{i}})$ implies $f(\mathbf{j}) < f(\mathbf{i})$, where $\mathbf{j}$ is the multi-index attached to the set $J = \hat{J} \cup (I \setminus \hat{I})$.

It remains to prove the assertion in the case where $\hat{I}$ is reverse symplectic. Here we need the preparations of this section. By the reverse symplectic condition applied to tableaux of shape ω_r we have $\sum_{i=j}^m \lambda_i \leqslant m - j + 1$ for all $j > 1$, where $(\lambda_1, \dots, \lambda_m) = f(\mathbf{i})$. Because I itself is non-reverse symplectic we must have $r = |I| = \sum_{i=1}^m \lambda_i > m$. According to Lemma 13.4 we conclude $v_I \in N$. But this implies the assertion of Proposition 12.1 in the remaining case too.

14. Proof of Proposition 12.2

In order to prove the proposition we have to consider generalizations of the elements $D_1, \dots, D_m$, which are defined for any $l \leqslant m$ by

$$D_{a,l} := \sum_{K \in P(l,a)} d_K.$$

For a positive integer k, define the y^{-1}-integer $\{k\}_{y^{-1}} := 1 + y^{-1} + y^{-2} + \dots + y^{-k+1} \in R$.

Lemma 14.1. *Let $a \in \underline{m}$. Then we have*

$$D_{1,l} D_{a,l} \equiv \{a+1\}_{y^{-1}} D_{a+1,l}$$

modulo the ideal spanned by D_1.

Proof. Using the above introduced notations we may write the right-hand side of (22) as

$$d_l^2 = (y-1) \sum_{i=l+1}^{m} d_l d_i.$$

Since $\sum_{i=l+1}^m d_i + d_l + D_{1,l-1} = D_1$, we deduce $d_l^2 \equiv (y^{-1}-1) d_l D_{1,l-1}$ modulo D_1 if $l > 1$ and $d_1^2 \equiv 0$ modulo D_1 in the case $l = 1$.

We proceed by induction on l. If $l = 1$, both sides are zero if $a > 1$. In the case $a = 1$ we have to show that $d_1^2 \equiv 0$ which was proved above.

For the induction step we write $D_{a,l} = d_l D_{a-1,l-1} + D_{a,l-1}$ and obtain

$$\begin{aligned} D_{1,l} D_{a,l} &= d_l^2 D_{a-1,l-1} + d_l D_{a,l-1} + D_{1,l-1}(d_l D_{a-1,l-1} + D_{a,l-1}) \\ &\equiv \big((y^{-1}-1)\{a\}_{y^{-1}} + 1 + \{a\}_{y^{-1}}\big) d_l D_{a,l-1} + \{a+1\}_{y^{-1}} D_{a+1,l-1}. \end{aligned}$$

Since $(y^{-1}-1)\{a\}_{y^{-1}} + 1 + \{a\}_{y^{-1}} = \{a+1\}_{y^{-1}}$, the lemma follows. $\square$

We introduce some new conventions. To an ordered subset $J = \{j_1, j_2, \dots, j_a\} \subseteq \underline{m}$ we define corresponding multi-indices by

$$\mathbf{j}^2_{\prec} := (j_1, j_1', j_2, j_2', \dots, j_a, j_a'), \qquad \mathbf{j}^2_{<} := (j_1, j_2, \dots, j_a, j_a', j_{a-1}', \dots, j_1').$$

Furthermore, we write

$$u_J := -\sum_{j \in J} j.$$

From the definition of d_J, we have $d_J = q^{u_J} v_{\mathbf{j}^2_\prec}$. By the relations of the exterior algebra there is another integer a_J such that $v_{\mathbf{j}^2_<} = q^{a_J} v_{\mathbf{j}^2_<}$. By Proposition 11.6 and Corollary 11.9 we calculate

$$\tau_\wedge(d_J) = \sum_{I \in P(n,2a)} q^{u_J + a_J} v_I \otimes T_q^{\omega_{2a}}\bigl(\mathbf{i} : \mathbf{j}^2_<\bigr) = \sum_{I \in P(n,2a)} q^{u_J} v_I \otimes T_q^{\omega_{2a}}\bigl(\mathbf{i} : \mathbf{j}^2_\prec\bigr).$$

Setting

$$G_{\mathbf{i},l,a} = \sum_{J \in P(l,a)} q^{u_J} T_q^{\omega_{2a}}\bigl(\mathbf{i} : \mathbf{j}^2_\prec\bigr),$$

we may write $\tau_\wedge(D_a) = \sum_{I \in P(n,2a)} v_I \otimes G_{\mathbf{i},m,a}$. Since the v_I form a free basis of the comodule the following proposition holds:

Proposition 14.2. *Proposition* 12.2 *holds if and only if* $G_{\mathbf{i},m,a} = 0$ *for all* $\mathbf{i} \in I(n,r)$ *and* $a \in \underline{m}$.

We will prove the equation $G_{\mathbf{i},m,a} = 0$ with the help of the Laplace expansion which is a special case of Laplace duality (Proposition 9.7) applied to the partitions

$$\lambda_t := (r - t + 1, \underbrace{1, 1, \ldots, 1}_{(t-1)\ \text{times}}) \in \Lambda^+(t,r),$$

where $1 \leqslant t \leqslant r$.

Caution. The symbol should not be confused with the tth component of a partition λ. A bideterminant $T_q^{\lambda_t}(\mathbf{i} : \mathbf{j})$ is the product of a $t \times t$ minor determinant with a monomial, that is

$$\begin{aligned} T_q^{\lambda_t}(\mathbf{i} : \mathbf{j}) &= T_q^{\omega_t}\bigl((i_1, \ldots, i_t) : (j_1, \ldots, j_t)\bigr) x_{i_{t+1} j_{t+1}} x_{i_{t+2} j_{t+2}} \ldots x_{i_r j_r} \\ &= \begin{vmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_t} \\ \vdots & & \vdots \\ x_{i_t j_1} & \cdots & x_{i_t j_t} \end{vmatrix}_q x_{i_{t+1} j_{t+1}} x_{i_{t+2} j_{t+2}} \ldots x_{i_r j_r}; \end{aligned}$$

in particular, $\lambda_r = \omega_r$.

Let L_t denote the set of distinguished left coset representatives of $S_{\lambda'_{t-1}}$ in $S_{\lambda'_t}$. Using basic transpositions s_i this set can be written down explicitly:

$$L_t = \{\mathrm{id}, s_{t-1}, s_{t-2}s_{t-1}, \ldots, s_1 s_2 \ldots s_{t-1}\}.$$

Setting

$$\mu_t := \sum_{w \in L_t} (-y)^{-l(w)} \beta(w)$$

the quantum symplectic (left) Laplace expansion deduced from Proposition 9.7 reads

Proposition 14.3 *(Laplace expansion). By use of the above introduced notation the following equation is valid*:

$$\mu_t \wr T_q^{\lambda_{t-1}}(\mathbf{i} : \mathbf{j}) = T_q^{\lambda_t}(\mathbf{i} : \mathbf{j}).$$

In the classical case and $t = r$ this turns out to be the familiar Laplace expansion. There is a very useful recursive calculation rule for the endomorphisms μ_t:

$$-y^{-1} \mu_t \beta_t = \mu_{t+1} - \mathrm{id}_{V^{\otimes r}}. \tag{23}$$

Before we state the fundamental lemma of this section we remind the reader of the addition of multi-indices, for example $\mathbf{j}^2_{\prec} + (kk') = (j_1, j_1', \ldots, j_a, j_a', k, k')$.

Lemma 14.4. *Let $l, a \in \underline{m}$ and $\mathbf{i} \in I(n, 2a)$. Then*

$$G_{\mathbf{i},l,a} = \sum_{J \in P(l,a-1),\, k \in \underline{l}} q^{u_J - k} T_q^{\lambda_{2a-1}}\big(\mathbf{i} : \mathbf{j}^2_{\prec} + (kk')\big) \wr \big(\mathrm{id} - y^{-a} \beta_{2a-1}\big).$$

Proof. First we treat the case where $a = 1$. Here we have $G_{\mathbf{i},l,1} = \sum_{j=1}^{l} q^{-j} T_q^{\omega_2}(\mathbf{i} : (jj'))$ by definition. Since $P(l, 0) = \emptyset$ the summation on the right-hand side of the lemma is over $k \in \underline{l}$ too. Furthermore,

$$T_q^{\lambda_1}\big(\mathbf{i} : (kk')\big) \wr \big(\mathrm{id} - y^{-1}\beta_1\big) = x_{i_1 k} x_{i_2 k'} \wr \big(\mathrm{id} - y^{-1}\beta_1\big) = T_q^{\omega_2}\big(\mathbf{i} : (kk')\big).$$

Thus both sides of the equation are identical.

For the general case we use induction on l. In the case $l = 1$ we necessarily have $a = 1$, which has been treated above. In order to prove the induction step we may assume $a > 1$ and $l > 1$. We divide the summation on the right-hand side into three subsums:

$$\text{(A) } l \in J, \qquad \text{(B) } l \notin J,\ k = l, \qquad \text{(C) } l \notin J,\ k < l,$$

and write $\sum_{\mathrm{A}}$, $\sum_{\mathrm{B}}$, and $\sum_{\mathrm{C}}$, respectively. First we treat subsum $\sum_{\mathrm{A}}$. Using Lemma 4.2 we see

$$\sum_{k \in \underline{l}} q^{-k} x_{i_{2a-1} k} x_{i_{2a} k'} \wr \beta = \sum_{k \in \underline{l}} q^{k-2l} x_{i_{2a-1} k'} x_{i_{2a} k}$$

and therefore

$$\sum_{k\in\underline{l}} q^{-k-l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk')\big) \wr \beta_{2a-1} = \sum_{k\in\underline{l}} q^{k-3l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'k)\big).$$

If $J \in P(l, a-1)$ contains l, we may write $J = \hat{J} \cup \{l\}$ with some $\hat{J} \in P(l-1, a-2)$ such that for the corresponding multi-index $\hat{\mathbf{j}}$ we have

$$T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk')\big) = T_q^{\lambda_{2a-1}}\big(\mathbf{i}:(\hat{\mathbf{j}})^{\mathbf{2}}_{\prec}+(ll'kk')\big).$$

We obtain:

$$\begin{aligned}
\sum_{\mathrm{A}} &= \sum_{J\in P(l-1,a-2),\, k\in\underline{l}} q^{u_J-k-l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'kk')\big) \wr \big(\mathrm{id}-y^{-a}\beta_{2a-1}\big) \\
&= \sum_{J\in P(l-1,a-2),\, k\in\underline{l-1}} q^{u_J-k-l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'kk')\big) \\
&\quad - y^{-a} q^{u_J+k-3l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'k'k)\big) \\
&\quad + q^{-2l}\Bigg(\sum_{J\in P(l-1,a-2)} q^{u_J} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'll')\big) \\
&\quad - y^{-a} q^{u_J} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'l'l)\big)\Bigg).
\end{aligned} \tag{24}$$

The bideterminant $T_q^{\lambda_{2a-1}}(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'l'l))$ vanishes by Corollary 9.2. Unfortunately the bideterminant with $(ll'll')$ is not zero in general. Since $D_1 \in N$ we have

$$(y-1)\sum_{k=l+1}^{m} d_k \equiv -(y-1)\sum_{k=1}^{l} d_k \mod N$$

and a routine calculation using relation (14) of the exterior algebra shows the congruence relation

$$d_l \equiv \big(y^{-1}-1\big)\sum_{k=1}^{l-1} d_k + y^{-l} c_l \mod N.$$

By Corollary 11.9 we obtain

$$T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'll')\big) = \big(y^{-1}-1\big) q^l \sum_{k=1}^{l-1} q^{-k} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big). \tag{25}$$

Note that the term involving c_l vanishes by Corollary 9.2. Since

$$\beta_{2a-1}\beta_{2a-2}(v_{\mathbf{j}^2_{\prec}+(k'kll')}) = q^2 v_{\mathbf{j}^2_{\prec}+(k'll'k)} \quad \text{and} \quad \beta^{-1}_{2a-1}\beta^{-1}_{2a-2}(v_{\mathbf{j}^2_{\prec}+(kk'll')}) = q^{-2} v_{\mathbf{j}^2_{\prec}+(kll'k')}$$

by (17) we may again deduce from Corollary 11.9 that

$$\begin{aligned} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'kk')\big) &= T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta^{-1}_{2a-1}\beta^{-1}_{2a-2}, \\ T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll'k'k)\big) &= T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'kll')\big) \wr \beta_{2a-1}\beta_{2a-2}. \end{aligned} \tag{26}$$

Here, in addition, we have used the equations $v_{(ll'k')} = q^2 v_{(k'll')}$ and $v_{(ll'k)} = q^{-2} v_{(kll')}$ which are valid inside the exterior algebra. Modulo the ideal $\mathcal{G}_{2a}$ of $\mathcal{A}_{2a}$ generated by γ the congruence relation $\beta^{-1} \equiv (y^{-1}\beta + (y^{-1}-1)\,\mathrm{id})$ holds by (1). Therefore, modulo this ideal the congruence

$$\beta^{-1}_{2a-1}\beta^{-1}_{2a-2} \equiv y^{-2}\beta_{2a-1}\beta_{2a-2} + y^{-1}\big(y^{-1}-1\big)(\beta_{2a-1}+\beta_{2a-2}) + \big(y^{-1}-1\big)^2\,\mathrm{id}$$

is valid which implies

$$\begin{aligned} &T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta^{-1}_{2a-1}\beta^{-1}_{2a-2} \\ &\quad = y^{-2} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta_{2a-1}\beta_{2a-2} \\ &\qquad + y^{-1}\big(y^{-1}-1\big) T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta_{2a-1} \\ &\qquad - \big(y^{-1}-1\big) T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \end{aligned} \tag{27}$$

by Lemma 9.3. Here we have also used the fact that $T_q^{\lambda_{2a-1}}(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')) \wr \beta_{2a-2} = -T_q^{\lambda_{2a-1}}(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll'))$ by Lemma 9.4 since $s_{2a-2} \in S_{\lambda'_{t-1}}$. Now substitute (27) into the first equation of (26) and Eqs. (26) and (25) into (24). Note that the terms coming from (25) and the last term of (27) cancel each other. We obtain the following expression for the subsum (A):

$$\begin{aligned} \sum_{\mathrm{A}} = \sum_{J\in P(l-1,a-2),\,k\in\underline{l-1}} \Big[& y^{-2} q^{u_J-k-l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta_{2a-1}\beta_{2a-2} \\ & - y^{-a} q^{u_J+k-3l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'kll')\big) \wr \beta_{2a-1}\beta_{2a-2} \\ & + \big(y^{-2}-y^{-1}\big) q^{u_J+k-l} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) \wr \beta_{2a-1} \Big]. \end{aligned} \tag{28}$$

Now we apply Laplace expansion (Proposition 14.3) twice to the first and second bideterminant and once to the third:

$$\begin{aligned} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) &= \mu_{2a-1}\mu_{2a-2}\wr T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big),\\ T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'kll')\big) &= \mu_{2a-1}\mu_{2a-2}\wr T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'kll')\big),\\ T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big) &= \mu_{2a-1}\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big). \end{aligned} \tag{29}$$

Note that we may commute the $\beta_{2a-1}\beta_{2a-2}$ from the right of $T_q^{\lambda_{2a-3}}(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll'))$ to the left by (5), since a bideterminant corresponding to the partition λ_{2a-3} is a monomial on the part where β_{2a-1} and β_{2a-2} are operating. A similar fact is true concerning the bideterminant $T_q^{\lambda_{2a-2}}(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll'))$ with respect to λ_{2a-2} and β_{2a-1}. Furthermore, note that by Lemma 4.2 for fixed J and arbitrary $\mathbf{i}\in I(n,r)$ we have

$$\sum_{k=1}^{l-1} q^{k-2(l-1)} T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(k'kll')\big) = \sum_{k=1}^{l-1} q^{-k} T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big)\wr\beta_{2a-3}.$$

Thus the second term of the right-hand side in (28) can be replaced by

$$y^{-(a-1)}y^{-2}q^{u_J-k-l}\mu_{2a-1}\mu_{2a-2}\beta_{2a-1}\beta_{2a-2}\wr T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big)\wr\beta_{2a-3},$$

since the summation from $k=1$ to $l-1$ equalizes both terms. Observe that after these operations the first and second term of (28) only differ by the factor $-y^{-(a-1)}\beta_{2a-3}$ so that these both summands can be factored by $(\mathrm{id}-y^{-(a-1)}\beta_{2a-3})$ (with respect to the $\wr$ symbol). Applying (29), substituting these equations into (28) and commuting β_{2a-1} and β_{2a-2} to the left of the corresponding bideterminant as explained above we obtain

$$\begin{aligned} \sum_{\mathrm{A}} = \sum_{J\in P(l-1,a-2),\,k\in\underline{l-1}} & q^{u_J-k-l} \\ &\times\Big[y^{-2}\mu_{2a-1}\mu_{2a-2}\beta_{2a-1}\beta_{2a-2}\wr T_q^{\lambda_{2a-3}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big)\wr\big(\mathrm{id}-y^{-(a-1)}\beta_{2a-3}\big)\\ &+y^{-1}\big(y^{-1}-1\big)\mu_{2a-1}\beta_{2a-1}\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big)\Big]. \end{aligned}$$

Note that the doubled use of $\wr$ is well defined by the last line of (3). To the first term of that sum we can apply the induction hypothesis. To this claim note that the symbol $\wr$ on the left of the bideterminant stands for a sum over the bideterminant's left multi-index. In order to apply the induction hypothesis this summation has to be commuted with the summation under the $\sum$-symbol. Similarly we can apply Lemma 14.1 together with Corollary 11.9 to the second term of the sum above in the following way: By Lemma 14.1 we can write $D_{a-2,l-1}D_{1,l-1}\equiv\{a-1\}_{y^{-1}}D_{a-1,l-1}$ modulo the ideal spanned by D_1. Therefore, by Corollary 11.9 together with Proposition 4.3 we have

$$\sum_{J\in P(l-1,a-2)}\sum_{k=1}^{l-1} q^{-k} T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(kk'll')\big)$$

$$= \sum_{J\in P(l-1,a-1)} \{a-1\}_{y^{-1}} T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll')\big)+D$$

for some $D\in\langle d_q\rangle$. Since $D=0$ inside $A^{\mathrm{sh}}_{R,q}(n)$ this results in

$$\sum_{\mathrm{A}} = \sum_{J\in P(l-1,a-1)} q^{u_J-l}\Big(y^{-2}\mu_{2a-1}\mu_{2a-2}\beta_{2a-1}\beta_{2a-2}\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll')\big)$$
$$+\,y^{-1}\big(y^{-1}-1\big)\{a-1\}_{y^{-1}}\mu_{2a-1}\beta_{2a-1}\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}+(ll')\big)\Big).$$

Since μ_{2a-2} and β_{2a-1} commute, we see from Eq. (23)

$$\begin{aligned} y^{-2}\mu_{2a-1}\mu_{2a-2}\beta_{2a-1}\beta_{2a-2} &= \big(-y^{-1}\mu_{2a-1}\beta_{2a-1}\big)\big(-y^{-1}\mu_{2a-2}\beta_{2a-2}\big)\\ &= (\mu_{2a}-\mathrm{id})(\mu_{2a-1}-\mathrm{id}). \end{aligned}$$

Similarly we calculate using (23) again

$$y^{-1}\big(y^{-1}-1\big)\{a-1\}_{y^{-1}}\mu_{2a-1}\beta_{2a-1} = \big(1-y^{-(a-1)}\big)(\mu_{2a}-\mathrm{id}).$$

Finally we obtain the following expression for the subsum (A):

$$\sum_{\mathrm{A}} = \sum_{J\in P(l,a),\, l\in J} q^{u_J}\big(\mu_{2a}\mu_{2a-1}-\mu_{2a-1}-y^{-(a-1)}(\mu_{2a}-\mathrm{id})\big)\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big).$$

The calculation of subsum (B) only needs one application of Laplace expansion together with the commutation of β_{2a-1} from the right of the bideterminant to the left and another application of (23):

$$\begin{aligned} \sum_{\mathrm{B}} &= \sum_{J\in P(l,a),\, l\in J} q^{u_J} T_q^{\lambda_{2a-1}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big)\wr\big(\mathrm{id}-y^{-a}\beta_{2a-1}\big)\\ &= \sum_{J\in P(l,a),\, l\in J} q^{u_J}\big(\mu_{2a-1}+y^{-(a-1)}(\mu_{2a}-\mathrm{id})\big)\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big). \end{aligned}$$

Thus subsum (A) and (B) together equal

$$\sum_{\mathrm{A}}+\sum_{\mathrm{B}} = \sum_{J\in P(l,a),\, l\in J} q^{u_J}\mu_{2a}\mu_{2a-1}\wr T_q^{\lambda_{2a-2}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big) = \sum_{J\in P(l,a),\, l\in J} q^{u_J} T_q^{\lambda_{2a}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big),$$

where for the second step Proposition 14.3 is used once more. To the subsum (C) the induction hypothesis can be applied directly:

$$\sum_{\mathrm{C}} = G_{\mathbf{i},l-1,a} = \sum_{J\in P(l-1,a)} q^{u_J} T_q^{\lambda_{2a}}\big(\mathbf{i}:\mathbf{j}^2_{\prec}\big).$$

Thus, it follows that all three subsums add up to $G_{\mathbf{i},l,a}$. □

Now we are able to prove Proposition 12.2. As we have seen in Proposition 14.2, we have to show $G_{\mathbf{i},m,a} = 0$ for $a = 1, \ldots, m$ and $\mathbf{i} \in I(n, 2a)$. From Proposition 4.3 we already know $G_{\mathbf{i},m,1} = 0$ for all $\mathbf{i} \in I(n, 2)$ since $d_q = 0$ inside $A^{\mathrm{sh}}_{R,q}(n)$. We will deduce the general case by induction on a with the help of Lemma 14.4. Let $a > 1$ and $\mathbf{i} \in I(n, 2a)$ be arbitrary. We apply Laplace expansion to the formula of the lemma:

$$G_{\mathbf{i},m,a} = \sum_{J \in P(m,a-1),\, k \in \underline{m}} q^{u_J - k} \mu_{2a-1} \wr T_q^{\lambda_{2a-2}}\big(\mathbf{i} : \mathbf{j}^2_{\prec} + (kk')\big) \wr \big(\mathrm{id} - y^{-a}\beta_{2a-1}\big).$$

As in the proof of the lemma we may commute $(\mathrm{id} - y^{-a}\beta_{2a-1})$ to the other side of the bideterminant. Let $(\mu_{\mathbf{ih}})_{\mathbf{i},\mathbf{h} \in I(n,2a)}$ be the coefficient matrix of the endomorphism $\mu_{2a-1}(\mathrm{id} - y^{-a}\beta_{2a-1})$ with respect to the canonical basis. We denote the multi-index consisting of the first $2a-2$ indices of $\mathbf{h}$ by $\bar{\mathbf{h}} := (h_1, \ldots, h_{2a-2})$ and obtain

$$\begin{aligned} G_{\mathbf{i},m,a} &= \sum_{\mathbf{h} \in I(n,2a)} \mu_{\mathbf{ih}} \sum_{J \in P(m,a-1)} q^{u_J} T_q^{\omega_{2a-2}}\big(\bar{\mathbf{h}} : \mathbf{j}^2_{\prec}\big) \sum_{k=1}^{m} q^{-k} x_{h_{2a-1}k} x_{h_{2a}k'} \\ &= \sum_{\mathbf{h} \in I(n,2a)} \mu_{\mathbf{ih}} G_{\bar{\mathbf{h}},m,a-1} \sum_{k=1}^{m} q^{-k} x_{h_{2a-1}k} x_{h_{2a}k'} = 0, \end{aligned}$$

since $G_{\bar{\mathbf{h}},m,a-1} = 0$ for all $\mathbf{h} \in I(n, 2a)$ by the induction hypothesis.

Remark 14.5. The proof of the classical version [O2, Proposition 8.2] of Proposition 12.2 relies on the embedding of the symplectic monoid $\mathrm{SpM}_n(R)$ into the ring $\mathrm{M}_n(R)$ of $(n \times n)$-matrices. Since in the quantum case such an embedding is missing the proof given here is absolutely incomparable to the one given in [O2, Proposition 8.2]. Additionally the statement of the latter proposition is a little bit stronger although this is not really needed. It is a good exercise to work out a classical version of the current section in order to understand the ideas of the proof. The result will be a simplification of the corresponding classical treatment in [O2].

Remark 14.6. Lemma 14.4 is the fundamental key to remove the restrictions from [O1, Basissatz 3.14.12 and Satz 4.1.2] in Theorems 7.1 and 7.3.

15. Finishing the proof of Theorem 7.1

Let us briefly recall what we have done so far. With respect to the proof that $\mathbf{B}_r$ is a basis, we have reduced the fact that it is a set of generators in Section 8 to the verification of Proposition 8.2, which we just have completed. Furthermore we already know from Section 6 that the axiom (C2*) of a cellular coalgebra is valid. It remains to show axiom (C3*) and the fact that $\mathbf{B}_r$ is linearly independent.

Let us start with the latter task. It is clearly enough to consider the case where $R = \mathcal{Z} = \mathbb{Z}[q, q^{-1}]$. Let $\mathbb{K}$ be the field of fractions of $\mathcal{Z}$ and let ϵ be the image of q

under the embedding of $\mathcal{Z}$ into $\mathbb{K}$. Any relation among elements of $\mathbf{B}_r$ with coefficients from $\mathcal{Z}$ is a relation with coefficients from the field $\mathbb{K}$ too. Thus, we only have to show $|\mathbf{B}_r| = \dim_{\mathbb{K}} A^{\mathrm{s}}_{\mathbb{K},\epsilon}(n,r)$. Now, $A^{\mathrm{s}}_{\mathbb{K},\epsilon}(n,r)$ is the centralizer coalgebra of the algebra $\mathcal{A}_r$ generated by the endomorphisms β_i and γ_i acting on $V^{\otimes r}$ in the sense of [O2, Section 2]. Consequently, by the comparison theorem [O2, Theorem 3.3] the dimension in question is the same as the dimension of the centralizer algebra of $\mathcal{A}_r$ acting on $V^{\otimes r}$.

The latter dimension can be deduced from well-known results from the theory of quantum groups. We will use [CP, Theorem 10.2.5 ii, second statement]. The operator called I^{ii+1}_{ϵ} there equals our $\epsilon^{-1}\beta_i$, thus $\epsilon I^{ii+1}_{\epsilon} = \beta_i$. The application of the theorem shows that the centralizer of our algebra $\mathcal{A}_r$ is identical to the image of the *quantized universal enveloping algebra* (QUE) corresponding to the Dynkin diagram C_m under its action on $V^{\otimes r}$. Now, by [CP, Proposition 10.1.13 and Theorem 10.1.14], the tensor space $V^{\otimes r}$ decomposes into irreducibles as a QUE-module because $\epsilon \in \mathbb{K}$ is transcendental over $\mathbb{Q}$. These irreducibles are indexed by the highest weights of the symplectic group and their dimensions are the same as in the classical case. The weights occurring are the same as for the symplectic group as well and correspond precisely to the elements of the set Λ from the definition of $\mathbf{B}_r$ (cf. [O2, 7.1]). It follows from work of R.C. King that the dimensions of the irreducibles are just $|M(\lambda)|$ [Ki] (cf. [Do2]). Consequently, we obtain the required identity:

$$\dim_{\mathbb{K}} A^{\mathrm{s}}_{\mathbb{K},\epsilon}(n,r) = \sum_{\lambda\in\Lambda} \bigl|M(\lambda)\bigr|^2 = |\mathbf{B}_r|.$$

Remark 15.1. The approach to the symplectic q-Schur algebra using the quantized universal enveloping algebra as outlined above has been investigated in [Dt2]. There, another cellular basis has been constructed (see [Dt2, 5.2 and 7.3]).

We now verify axiom (C3*). We abbreviate

$$\mathcal{K} := A^{\mathrm{s}}_{R,q}(n,r).$$

Let $D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}} \in \mathbf{B}_r$ where $\underline{\lambda} := (\lambda, l) \in \Lambda$ and $\mathbf{i},\mathbf{j} \in M(\lambda)$. As d^l_q is grouplike and Δ a homomorphism of algebras we calculate using (20) that

$$\Delta\bigl(D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}}\bigr) = \bigl(d^l_q \otimes d^l_q\bigr)\Delta\bigl(T^{\lambda}_q(\mathbf{i}:\mathbf{j})\bigr) = \sum_{\mathbf{h}\in I^<_{\lambda}} D^{\underline{\lambda}}_{\mathbf{i},\mathbf{h}} \otimes D^{\underline{\lambda}}_{\mathbf{h},\mathbf{j}}.$$

Here, as in Section 11, $I^<_{\lambda}$ is the set of multi-indices that are λ-column-standard with respect to the usual order $<$ on $\underline{n}$ (see Section 8). Now, according to the straightening formula 8.2 (after application of *) to each $\mathbf{h} \in I^<_{\lambda}$ and $\mathbf{k} \in M(\lambda)$ there is an element $a_{\mathbf{hk}} \in R$ (unique by the linear independence of $\mathbf{B}_r$) such that

$$D^{\underline{\lambda}}_{\mathbf{h},\mathbf{j}} \equiv \sum_{\mathbf{k}\in M(\lambda)} a_{\mathbf{hk}} D^{\underline{\lambda}}_{\mathbf{k},\mathbf{j}} \mod \mathcal{K}(>\underline{\lambda}). \tag{30}$$

We set

$$h(\mathbf{k},\mathbf{i}) := \sum_{\mathbf{h}\in I_\lambda^<} D^{\underline{\lambda}}_{\mathbf{i},\mathbf{h}} a_{\mathbf{hk}} \in \mathcal{K}(\geqslant \underline{\lambda})$$

and obtain

$$\Delta\big(D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}}\big) \equiv \sum_{\mathbf{k}\in M(\lambda)} h(\mathbf{k},\mathbf{i}) \otimes D^{\underline{\lambda}}_{\mathbf{k},\mathbf{j}} \mod \mathcal{K}(\geqslant \underline{\lambda}) \otimes \mathcal{K}(> \underline{\lambda}).$$

This completes the verification of axiom (C3*) and hence the proof of Theorem 7.1.

16. Quasi-heredity of the symplectic q-Schur algebra

In [GL] Graham and Lehrer have presented a nice criterion for quasi-heredity of a cellular algebra which we will now verify in our case. This will prove Theorem 7.5.

To this aim we have to investigate the bilinear form $\phi_{\underline{\lambda}}$ on the standard modules $W(\underline{\lambda})$. We must show that they are not zero [GL, 3.10]. Let us first calculate the Gram matrix of $\phi_{\underline{\lambda}}$ with respect to the basis $\{C^{\underline{\lambda}}_{\mathbf{i}} \mid \mathbf{i} \in M(\underline{\lambda})\}$ of $W(\underline{\lambda})$. We abbreviate its entries by

$$\phi_{\mathbf{ij}} := \phi_{\underline{\lambda}}\big(C^{\underline{\lambda}}_{\mathbf{i}}, C^{\underline{\lambda}}_{\mathbf{j}}\big) \in R.$$

According to the definition in [GL, 2.3], these numbers satisfy

$$C^{\underline{\lambda}}_{\mathbf{i},\mathbf{k}} C^{\underline{\lambda}}_{\mathbf{l},\mathbf{j}} \equiv \phi_{\mathbf{kl}} C^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}} \mod S^{\mathrm{s}}_{R,q}(n,r)(< \underline{\lambda}).$$

Such a congruence relation is valid with $\phi_{\mathbf{kl}}$ being independent of $\mathbf{i}$ and $\mathbf{j}$ by the axioms of a cellular algebra (see [GL, 1.7]). Dualizing this congruence we obtain the following counterpart in the cellular coalgebra $\mathcal{K} = A^{\mathrm{s}}_{R,q}(n,r)$:

$$\Delta\big(D^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}}\big) \equiv \sum_{\mathbf{k},\mathbf{l}\in M(\underline{\lambda})} \phi_{\mathbf{kl}} D^{\underline{\lambda}}_{\mathbf{i},\mathbf{k}} \otimes D^{\underline{\lambda}}_{\mathbf{l},\mathbf{j}}$$

modulo $\mathcal{K}(\geqslant \underline{\lambda}) \otimes \mathcal{K}(> \underline{\lambda}) + \mathcal{K}(> \underline{\lambda}) \otimes \mathcal{K}(\geqslant \underline{\lambda})$. According to the calculations for the verification of axiom (C3*) in the previous section we see using the notations from there that

$$\phi_{\mathbf{kl}} = \sum_{\mathbf{h}\in I_\lambda^<} a_{\mathbf{hk}} a_{\mathbf{hl}}. \tag{31}$$

The bilinear form $\phi_{\underline{\lambda}}$ is different from zero if this is the case for a single entry $\phi_{\mathbf{kl}}$. We calculate $\phi_{\mathbf{kk}}$ where $\mathbf{k}$ is the λ-tableau $T^\lambda_{\mathbf{k}} = T$ with constant rows $T(i,j) := m+i$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant \lambda_j$. Obviously T is a standard tableau with respect to both orders on $\underline{n}$, namely $<$ as well as $\prec$. Furthermore the reverse symplectic condition holds because

$T(i,j) = (m-i+1)'$. Consequently we have $\mathbf{k} \in M(\underline{\lambda}) \cap I_\lambda^<$. Note that $\mathbf{k}$ does not contain any pair of associated indices (i,i'). The content $\eta := |\mathbf{k}|$ of $\mathbf{k}$ is given by

$$\eta_i = \begin{cases} 0, & i \leqslant m, \\ \lambda_{i-m}, & i > m. \end{cases}$$

Consider the endomorphism

$$\tau = \sum_{\mathbf{i} \in I(n,r),\, |\mathbf{i}| = \eta} e_{\mathbf{ii}} \in \mathrm{End}_R\left(V^{\otimes r}\right).$$

It is easy to see that τ commutes with β_i and γ_i for all $i = 1, \dots, r-1$. Consequently it is an endomorphism of the $A^{\mathrm{s}}_{R,q}(n,r)$ comodule $V^{\otimes r}$ (in fact it is the idempotent of $S^{\mathrm{s}}_{R,q}(n,r)$ corresponding to the weight space with weight η). There is an induced action of τ on $A^{\mathrm{s}}_{R,q}(n,r)$ from the left defined by

$$\tau x_{\mathbf{ij}} := (\tau \otimes \mathrm{id})\Delta(x_{\mathbf{ij}}) = \begin{cases} 0, & |\mathbf{i}| \neq \eta, \\ x_{\mathbf{ij}}, & |\mathbf{i}| = \eta. \end{cases}$$

For a bideterminant we have

$$\tau T_q^\lambda(\mathbf{i}:\mathbf{j}) = \begin{cases} 0, & |\mathbf{i}| \neq \eta, \\ T_q^\lambda(\mathbf{i}:\mathbf{j}), & |\mathbf{i}| = \eta. \end{cases}$$

Applying τ to the defining equation (30) of $a_{\mathbf{hk}}$ we see that $a_{\mathbf{hk}} = 0$ if $|\mathbf{h}| \neq \eta$ by linear independence of $D^{\underline{\lambda}}_{\mathbf{k},\mathbf{j}}$. Since $\mathbf{k}$ is the only element in $I_\lambda^< \cap M(\lambda)$ having content η it follows that $a_{\mathbf{hk}} = 0$ if $\mathbf{h} \neq \mathbf{k}$ and we conclude

$$\phi_{\mathbf{kk}} = \sum_{\mathbf{h} \in I_\lambda^<} a_{\mathbf{hk}}^2 = a_{\mathbf{kk}}^2 = 1.$$

By [GL, Remark 3.10], this finishes the proof of Theorem 7.5.

17. Outlook

Dualizing the coalgebra map

$$A^{\mathrm{s}}_{R,q}(n,r-2) \xrightarrow{\cdot d_q} A^{\mathrm{s}}_{R,q}(n,r)$$

from the sequence (9), one obtains an epimorphism of algebras from $S_q^s(n,r)$ to $S_q^s(n,r-2)$. On a basis element $C^{\underline{\lambda}}_{\mathbf{i},\mathbf{j}}$ it is given by subtracting 1 from l in $\underline{\lambda} = (\lambda, l)$ and keeping $\mathbf{i}, \mathbf{j}$ fixed. Its kernel is the linear span of those basis elements which occur in the case $l = 0$. This forces a recursive structure on the representation theory of these algebras in a similar way as is known for the Birman–Murakami–Wenzl algebras (see [BW]).

In addition these epimorphisms can be used to define an inverse limit of the symplectic q-Schur algebras in a similar way as has been worked out for the type A q-Schur algebra in [GR, Section 6.4]. It seems to be plausible that accordingly the quantized universal enveloping algebra embeds into this inverse limit (cf. [Dt2, 7.3]).

Concerning analogues to the orthogonal case, note that Lemma 11.7 will not work here. Maybe, a way out is to consider coefficient functions of the symmetric algebra, i.e. the elements

$$\sum_{w\in\mathcal{S}_\lambda} y^{-l(w)}\beta(w)\wr x_{\mathbf{ij}} = \sum_{w\in\mathcal{S}_\lambda} y^{-l(w)}x_{\mathbf{ij}}\wr\beta(w)$$

instead of bideterminants, which are coefficient functions of the exterior algebra.

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Appendix A. Technical details

A.1. Details to the proof of Proposition 11.2

With the help of the Diamond Lemma for Ring Theory [Bg] we will construct a free basis for $\bigwedge_{R,q}(n)$. Our order $\prec$ on $\underline{n}$ induces a lexicographic order on multi-indices $\mathbf{i}\in I(n,r)$ and on the corresponding monomials $v_{\mathbf{i}}\in V^{\otimes r}$, which will be denoted by the same symbol $\prec$. On the monomials of $\mathcal{T}(V)$ we get an induced partial order if we declare monomials of different degree to be incomparable. It is clear that $\prec$ is compatible with the semigroup structure of the set of monomials of $\mathcal{T}(V)$ as required in [Bg].

We now introduce a system of reductions of degree two in $\mathcal{T}(V)$ extracted from the relations (13), (14), and (16) of the exterior algebra and divide them accordingly into three types. As in [Bg] we write them as pairs consisting of a monomial and a substitution expression:

$$\text{(R1)}\quad \bigl(v_iv_j, -q^{\operatorname{sign}(j-i)}v_jv_i\bigr) \qquad \text{if } j\prec i,\ i\neq j',$$

$$\text{(R2)}\quad \left(v_{i'}v_i, -q^{-2}v_iv_{i'} - \bigl(q^{-2}-1\bigr)\sum_{j=i+1}^{m} q^{i-j}v_jv_{j'}\right) \qquad \text{if } 1\leqslant i\leqslant m,$$

$$\text{(R3)}\quad (v_iv_i, 0) \qquad \text{if } 1\leqslant i\leqslant n.$$

Since all monomials of the reduction system are greater than the monomials in the corresponding substitution expressions our partial order $\prec$ on $\mathcal{T}(V)$ is compatible with the reduction system.

The set of monomials in $V^{\otimes r}$ which do not contain any monomial of the reduction system as a subexpression clearly is

$$F_r := \{v_{\mathbf{i}} \mid \mathbf{i} = (i_1, \dots, i_r) \in I(n,r),\ i_1 \prec \dots \prec i_r\}.$$

Obviously F_r generates the rth homogeneous summand $\bigwedge_{R,q}(n,r)$ of the exterior algebra as an R-module. To see that these sets are linearly independent we must show that all ambiguities of the reduction system are resolvable. Since all monomials are of degree two only overlap ambiguities occur and we can reduce to the case of degree three. Ambiguities between reductions of type (R3) are trivially resolvable and such ones where both reductions are of type (R2) do not occur. Thus we have to handle the following remaining cases:

1. Both reductions are of type (R1):
   $v_i v_j v_k$ where $k \prec j \prec i$ and $i \neq j' \neq k$.
2. Ambiguities between (R1) and (R3):
   (a) $v_i v_i v_j$ where $j \prec i$ and $i \neq j'$,
   (b) $v_j v_i v_i$ where $i \prec j$ and $i \neq j'$.
3. Ambiguities between (R2) and (R3):
   (a) $v_{i'} v_i v_i$ where $1 \leqslant i \leqslant m$,
   (b) $v_{i'} v_{i'} v_i$ where $1 \leqslant i \leqslant m$.
4. Ambiguities between (R1) and (R2):
   (a) $v_i v_{j'} v_j$ where $j' \prec i$ and $1 \leqslant j \leqslant m$,
   (b) $v_{j'} v_j v_i$ where $i \prec j$ and $1 \leqslant j \leqslant m$.

In order to prove resolvability of these ambiguities we will write an application of a reduction as: monomial $\mapsto$ substitution expression. The first case can be solved in the following way beginning with reduction of the left-hand side pair

$$\begin{aligned} v_i v_j v_k &\mapsto -q^{\operatorname{sign}(j-i)} v_j v_i v_k \mapsto q^{\operatorname{sign}(j-i)+\operatorname{sign}(k-i)} v_j v_k v_i \\ &\mapsto -q^{\operatorname{sign}(j-i)+\operatorname{sign}(k-i)+\operatorname{sign}(k-j)} v_k v_j v_i, \end{aligned}$$

and then beginning with the right-hand side pair

$$\begin{aligned} v_i v_j v_k &\mapsto -q^{\operatorname{sign}(k-j)} v_i v_k v_j \mapsto q^{\operatorname{sign}(k-i)+\operatorname{sign}(k-j)} v_k v_i v_j \\ &\mapsto -q^{\operatorname{sign}(j-i)+\operatorname{sign}(k-i)+\operatorname{sign}(k-j)} v_k v_j v_i. \end{aligned}$$

The treatment of case 2 is very easy and does not need to be written down. In order to treat case 3(a) we have to show that starting with a reduction of type (R2) on the left-hand side pair finally reduces to zero:

$$v_{i'} v_i v_i \mapsto -q^{-2} v_i v_{i'} v_i - (q^{-2} - 1) \sum_{j=i+1}^{m} q^{i-j} v_j v_{j'} v_i$$

$$\mapsto -q^{-2}\left(-q^{-2}v_i v_i v_{i'} - \left(q^{-2}-1\right)\sum_{j=i+1}^{m} q^{i-j} v_i v_j v_{j'}\right)$$
$$- \left(q^{-2}-1\right)\sum_{j=i+1}^{m} q^{i-j+\operatorname{sign}(i-j')} v_j v_i v_{j'}$$
$$\mapsto q^{-2}\left(q^{-2}-1\right)\sum_{j=i+1}^{m} q^{i-j} v_i v_j v_{j'}$$
$$- \left(q^{-2}-1\right)\sum_{j=i+1}^{m} q^{i-j+\operatorname{sign}(i-j')+\operatorname{sign}(i-j)} v_i v_j v_{j'} = 0.$$

Part (b) of case 3 is similar and we can proceed to case 4. Condition $j' \prec i$ means that $i < j$ (the case $i = j$ has been treated above) or $i > j'$ whereas $i \prec j$ means that $j < i < j'$. Note that in general $a \prec b$ implies $a \prec b'$. As above we reduce beginning with the left-hand side pair in (a):

$$v_i v_{j'} v_j \mapsto \cdots \mapsto -q^{-2+\operatorname{sign}(j'-i)+\operatorname{sign}(j-i)} v_j v_{j'} v_i$$
$$- \left(q^{-2}-1\right)\sum_{k=j+1}^{m} q^{j-k+\operatorname{sign}(j'-i)+\operatorname{sign}(j-i)} v_k v_{k'} v_i$$

and then beginning with the right-hand side pair

$$v_i v_{j'} v_j \mapsto \cdots \mapsto -q^{-2+\operatorname{sign}(j'-i)+\operatorname{sign}(j-i)} v_j v_{j'} v_i$$
$$- \left(q^{-2}-1\right)\sum_{k=j+1}^{m} q^{j-k+\operatorname{sign}(k'-i)+\operatorname{sign}(k-i)} v_k v_{k'} v_i.$$

Since $j < k$ and in addition $i < j$ or $i > j'$ we have

$$\operatorname{sign}(j'-i) = \operatorname{sign}(k'-i) \quad \text{and} \quad \operatorname{sign}(j-i) = \operatorname{sign}(k-i).$$

Thus both reductions lead to the same expression. Turning to part (b) the calculation of both reductions lead to similar expressions but we have to divide the sum into a $i \prec k$ and a $k \prec i$ section. First we begin with the left-hand side pair in (b)

$$v_{j'} v_j v_i \mapsto \cdots \mapsto -q^{-2+\operatorname{sign}(i-j')+\operatorname{sign}(i-j)} v_i v_j v_{j'}$$
$$- \left(q^{-2}-1\right)\sum_{k=j+1,\, i\prec k}^{m} q^{j-k+\operatorname{sign}(i-j')+\operatorname{sign}(i-j)} v_i v_k v_{k'}$$
$$- \left(q^{-2}-1\right)\sum_{k=j+1,\, k\prec i}^{m} q^{j-k+\operatorname{sign}(k'-i)+\operatorname{sign}(k-i)} v_k v_{k'} v_i$$

and then beginning with the right-hand side pair

$$\begin{aligned} v_{j'}v_jv_i \mapsto \cdots \mapsto -q^{-2+\operatorname{sign}(i-j')+\operatorname{sign}(i-j)}v_iv_jv_{j'} \\ -\left(q^{-2}-1\right)\sum_{k=j+1,\,i\prec k}^{m} q^{j-k}v_iv_kv_{k'} \\ -\left(q^{-2}-1\right)\sum_{k=j+1,\,k\prec i}^{m} q^{j-k+\operatorname{sign}(k'-i)+\operatorname{sign}(k-i)}v_kv_{k'}v_i. \end{aligned}$$

Since $j<i<j'$ the expression $\operatorname{sign}(i-j)+\operatorname{sign}(i-j')$ is always zero. Thus both reductions coincide and the proof is finished.

A.2. Details to the proof of Proposition 11.3

We have to verify the third equation listed in the proof. Let us first find suitable expressions for $\beta(c_i)$ and $\beta(d_i)$.

$$\begin{aligned} \beta(c_i) &= q^i\beta(v_{i'}v_i) \\ &= q^i\left(v_iv_{i'}+(y-1)v_{i'}v_i-(y-1)\sum_{k=i+1}^{n} q^{\rho_k-\rho_i}\epsilon_k\epsilon_iv_kv_{k'}\right) \\ &= -y^id_i+(y-1)c_i-(y-1)q^i\left(\sum_{k=i+1}^{m} q^{i-k}v_kv_{k'}+\sum_{k=1}^{m}-q^{-\rho_k-\rho_i}v_{k'}v_k\right) \\ &= -y^id_i+(y-1)\left(c_i+y^i\left(\sum_{k=i+1}^{m} d_k+y^{-m-1}\sum_{k=1}^{m} c_k\right)\right), \\ \beta(d_i) &= -q^{-i}\beta(v_iv_{i'}) \\ &= -q^{-i}\left(v_{i'}v_i-(y-1)\sum_{k=i'+1}^{n} q^{\rho_k+\rho_{i'}}\epsilon_k\epsilon_{i'}v_kv_{k'}\right) \\ &= -y^{-i}c_i+(y-1)q^{-i}\sum_{k=1}^{i-1} q^{\rho_{k'}-\rho_{i'}}v_{k'}v_k \\ &= -y^{-i}c_i+(y-1)y^{-i}\sum_{k=1}^{i-1} c_k. \end{aligned}$$

Setting $d=y^{-i}c_i-y^{-1}d_i-(y^{-1}-1)\sum_{j=i+1}^{m} d_j$, we obtain

$$
\begin{aligned}
\beta(d) = {} & -d_i + y^{-i}(y-1)c_i + (y-1)\left(\sum_{k=i+1}^{m} d_k + y^{-m-1}\sum_{k=1}^{m} c_k\right) \\
& + y^{-i-1}c_i + y^{-i}\left(y^{-1}-1\right)\sum_{k=1}^{i-1} c_k \\
& - \left(y^{-1}-1\right)\sum_{j=i+1}^{m}\left(-y^{-j}c_j + y^{-j}(y-1)\sum_{k=1}^{j-1} c_k\right).
\end{aligned}
\tag{A.1}
$$

Let us focus attention on the summand displayed in the last line:

$$
\sum_{j=i+1}^{m}\left(-y^{-j}c_j + y^{-j}(y-1)\sum_{k=1}^{j-1} c_k\right) = -\left(\sum_{k=i+1}^{m} y^{-k}c_k\right) + (y-1)\sum_{j=i+1}^{m}\sum_{k=1}^{j-1} y^{-j}c_k.
$$

The second summand in this expression can be transformed in the following way:

$$
\begin{aligned}
\sum_{j=i+1}^{m}\sum_{k=1}^{j-1} y^{-j}c_k &= \sum_{j=i+1}^{m}\left(\sum_{k=1}^{i-1} y^{-j}c_k + \sum_{k=i}^{j-1} y^{-j}c_k\right) \\
&= \sum_{k=1}^{i-1}\left(\sum_{j=i+1}^{m} y^{-j}\right)c_k + \sum_{k=i}^{m-1}\left(\sum_{j=k+1}^{m} y^{-j}\right)c_k.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
(y-1)\sum_{j=i+1}^{m}\sum_{k=1}^{j-1} y^{-j}c_k &= \sum_{k=i}^{m-1} y^{-k}c_k - y^{-m}\sum_{k=i}^{m-1} c_k - y^{-m}\sum_{k=1}^{i-1} c_k + y^{-i}\sum_{k=1}^{i-1} c_k \\
&= \sum_{k=i}^{m-1} y^{-k}c_k - y^{-m}\sum_{k=1}^{m-1} c_k + y^{-i}\sum_{k=1}^{i-1} c_k \\
&= \sum_{k=i}^{m} y^{-k}c_k - y^{-m}\sum_{k=1}^{m} c_k + y^{-i}\sum_{k=1}^{i-1} c_k.
\end{aligned}
$$

In order to get the last line one has to add $y^{-m}c_m - y^{-m}c_m$. Substituting this result into Eq. (A.1) yields

$$
\begin{aligned}
\beta(d) = {} & -d_i + y^{-i+1}c_i - y^{-i}c_i + (y-1)\sum_{k=i+1}^{m} d_k + y^{-m-1}(y-1)\sum_{k=1}^{m} c_k \\
& + y^{-i-1}c_i + y^{-i}\left(y^{-1}-1\right)\sum_{k=1}^{i-1} c_k + \left(y^{-1}-1\right)\sum_{k=i+1}^{m} y^{-k}c_k
\end{aligned}
$$

$$-\,y^{-1}(y-1)\left(y^{-m}\sum_{k=1}^{m}c_k - y^{-i}\sum_{k=1}^{i-1}c_k - \sum_{k=i}^{m}y^{-k}c_k\right).$$

Now we see that almost all summands in this expression cancel each other and we end up with

$$\beta\left(y^{-i}c_i - y^{-1}d_i - \left(y^{-1}-1\right)\sum_{j=i+1}^{m}d_j\right) = y^{-i+1}c_i - d_i + (y-1)\sum_{j=i+1}^{m}d_j.$$

A.3. Details to the proof of Lemma 13.1

We prove a more general statement concerning elements $D'_{a,l}$ defined similar to the elements $D_{a,l}$ of Section 14. For the set of subsets $K \subseteq \underline{m} \setminus \underline{l-1}$ that have a elements we will write $P'(l,a)$. We set

$$D'_{a,l} := \sum_{K \in P'(l,a)} d_K.$$

We will need the following analogue to Lemma 14.1:

Lemma A.1. *Let $a \in \underline{m}$. Then we have*

$$D'_{1,l}D'_{a,l} = \{a+1\}_y D'_{a+1,l},$$

where $\{k\}_y := 1 + y + y^2 + \cdots + y^{k-1} \in R$.

Proof. We will prove this by induction on $m-l$. If $l=m$, both sides are zero if $a>1$. In the case $a=1$ we have to show that $d_m^2 = 0$ which follows by (22).

For the induction step we write

$$D'_{a,l} = d_l D'_{a-1,l+1} + D'_{a,l+1} \quad \text{and} \quad D'_{1,l} = d_l + D'_{1,l+1}.$$

Note that $d_l^2 = d_l(y-1)D'_{1,l+1}$. We obtain

$$\begin{aligned} D'_{1,l}D'_{a,l} &= d_l^2 D'_{a-1,l+1} + d_l D'_{a,l+1} + D'_{1,l+1}(d_l D'_{a-1,l+1} + D'_{a,l+1}) \\ &= \big((y-1)\{a\}_y + 1 + \{a\}_y\big)d_l D'_{a,l+1} + \{a+1\}_y D'_{a+1,l+1}. \end{aligned}$$

Since $(y-1)\{a\}_y + 1 + \{a\}_y = \{a+1\}_y$, the lemma follows. □

The more general statement of Lemma 13.1 reads: Let $a, l \in \underline{m}$. If $\underline{m} \setminus \underline{l-1} = L \cup M$ is a partition of $\underline{m} \setminus \underline{l-1}$ into disjoint subsets L and M then to each $K \in P'(l,a)$ there is an integer $s(K,L,l)$ such that

$$D'_{a,l} = \sum_{K \in P'(l,a)} y^{s(K,L,l)} c_{K\cap L} d_{K\cap M}.$$

We will prove this by induction on $m-l$ as well. If $m=l$ and $a>1$ then $D'_{a,m}=0$ and there is no $K\in P'(m,a)$. If $a=1$ we have $D'_{1,m}=d_m$ and $K=\{m\}$. Thus $s(K,\emptyset,m)=0$ and $s(K,\{m\},m)=1-m$ leads to a solution.

For the induction step we first consider the case $l\in M$ and calculate

$$\begin{aligned} D'_{a,l} &= d_l D'_{a-1,l+1} + D'_{a,l+1} \\ &= d_l \sum_{K\in P'(l+1,a-1)} y^{s(K,L,l+1)} c_{K\cap L} d_{K\cap M} + \sum_{K\in P'(l+1,a)} y^{s(K,L,l+1)} c_{K\cap L} d_{K\cap M} \\ &= \sum_{K\in P'(l,a),\, l\in K} y^{s(K\setminus\{l\},L,l+1)} c_{K\cap L} d_{K\cap M} + \sum_{K\in P'(l,a),\, l\notin K} y^{s(K,L,l+1)} c_{K\cap L} d_{K\cap M}. \end{aligned}$$

Setting $s(K,L,l):=s(K\setminus\{l\},L,l+1)$ leads to a solution. If $l\in L$ we apply relation (14) of the exterior algebra to obtain

$$\begin{aligned} D'_{a,l} &= d_l D'_{a-1,l+1} + D'_{a,l+1} \\ &= y^{1-l} c_l D'_{a-1,l+1} + (y-1) D'_{1,l+1} D'_{a-1,l+1} + D'_{a,l+1}. \end{aligned}$$

From Lemma A.1 we see that $D'_{1,l+1}D'_{a-1,l+1}=\{a\}_y D'_{a,l+1}$. Since $(y-1)\{a\}_y+1=y^a$ we obtain

$$D'_{a,l} = y^{1-l} c_l D'_{a-1,l+1} + y^a D'_{a,l+1}.$$

Thus, setting $s(K,L,l):=1-l+s(K\setminus\{l\},L\setminus\{l\},l+1)$ if $l\in K$ and $s(K,L,l):=a+s(K,L\setminus\{l\},l+1)$ otherwise leads to a solution.

It remains to check that $s(K,L,1)=v(K,L)$ if $K\subseteq M$. More generally we prove that

$$s(K,L,l) = (1-l)a + v(K,\underline{m}\setminus M)$$

by induction on $m-l$ again. If $l=m$ we must have $K=\{m\}=M$ and $L=\emptyset$. In this case both sides of the equation equal zero. For the induction step let us first consider the case $l\in K$. By the above calculation this gives

$$s(K,L,l) = s\big(K\setminus\{l\},L,l+1\big) = \big(1-(l+1)\big)(a-1) + v\big(K\setminus\{l\},\underline{m}\setminus M\cup\{l\}\big).$$

Since $v(K\setminus\{l\},\underline{m}\setminus M\cup\{l\}) = v(K,\underline{m}\setminus M)+a-l$ the assertion follows in the first case. Next we consider $l\in M\setminus K$. Here we have

$$s(K,L,l) = s(K,L,l+1) = \big(1-(l+1)\big)a + v\big(K,\underline{m}\setminus M\cup\{l\}\big)$$

and the assertion follows since $v(K,\underline{m}\setminus M\cup\{l\}) = v(K,\underline{m}\setminus M)+a$. Finally we have to consider $l\in L$. From the calculation above we get

$$s(K,L,l) = a + s\big(K,L\setminus\{l\},l+1\big) = a + \big(1-(l+1)\big)a + v(K,\underline{m}\setminus M),$$

which directly gives the result.

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